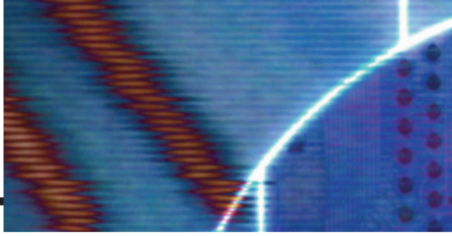


Appendix D

DETERMINANTS

Determinants are used to find the mathematical solutions for the variables in two or more simultaneous equations. Once the procedure is properly understood, solutions can be obtained with a minimum of time and effort and usually with fewer errors than when using other methods.

Consider the following equations, where x and y are the unknown variables and a_1, a_2, b_1, b_2, c_1, and c_2 are constants:

$$\begin{array}{ccccc} \text{Col. 1} & & \text{Col. 2} & & \text{Col. 3} \\ \hline a_1x & + & b_1y & = & c_1 \\ a_2x & + & b_2y & = & c_2 \end{array} \qquad \begin{array}{r} \textbf{(D.1a)} \\ \textbf{(D.1b)} \end{array}$$

It is certainly possible to solve for one variable in Eq. (D.1a) and substitute into Eq. (D.1b). That is, solving for x in Eq. (D.1a),

$$x = \frac{c_1 - b_1y}{a_1}$$

and substituting the result in Eq. (D.1b),

$$a_2\left(\frac{c_1 - b_1y}{a_1}\right) + b_2y = c_2$$

It is now possible to solve for y, since it is the only variable remaining, and then substitute into either equation for x. This is acceptable for two equations, but it becomes a very tedious and lengthy process for three or more simultaneous equations.

Using determinants to solve for x and y requires that the following formats be established for each variable:

$$\begin{array}{cc} \begin{array}{cc}\text{Col.} & \text{Col.} \\ 1 & 2\end{array} & \begin{array}{cc}\text{Col.} & \text{Col.} \\ 1 & 2\end{array} \\ \hline x = \dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} & y = \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \end{array} \qquad \textbf{(D.2)}$$

First note that only constants appear within the vertical brackets and that the denominator of each is the same. In fact, the denominator is simply the coefficients of x and y in the same arrangement as in Eqs. (D.1a) and (D.1b). When solving for x, replace the coefficients of x in the numerator by the constants to the right of the equal sign in Eqs. (D.1a) and (D.1b), and repeat the coefficients of the y variable. When solving for y, replace the y coefficients in the numerator by the constants to the right of the equal sign, and repeat the coefficients of x.

Each configuration in the numerator and denominator of Eq. (D.2) is referred to as a *determinant (D),* which can be evaluated numerically in the following manner:

$$\text{Determinant} = D = \begin{array}{c} \begin{array}{cc}\text{Col.} & \text{Col.} \\ \underline{1} & \underline{2}\end{array} \\ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{array} = a_1b_2 - a_2b_1 \qquad \textbf{(D.3)}$$

The expanded value is obtained by first multiplying the top left element by the bottom right and then subtracting the product of the lower left and upper right elements. This particular determinant is referred to as a *second-order* determinant, since it contains two rows and two columns.

It is important to remember when using determinants that the columns of the equations, as indicated in Eqs. (D.1a) and (D.1b), must be placed in the same order within the determinant configuration. That is, since a_1 and a_2 are in column 1 of Eqs. (D.1a) and (D.1b), they must be in column 1 of the determinant. (The same is true for b_1 and b_2.)

Expanding the entire expression for x and y, we have the following:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} \quad \textbf{(D.4a)}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1} \quad \textbf{(D.4b)}$$

EXAMPLE D.1 Evaluate the following determinants:

a. $\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = (2)(4) - (3)(2) = 8 - 6 = \mathbf{2}$

b. $\begin{vmatrix} 4 & -1 \\ 6 & 2 \end{vmatrix} = (4)(2) - (6)(-1) = 8 + 6 = \mathbf{14}$

c. $\begin{vmatrix} 0 & -2 \\ -2 & 4 \end{vmatrix} = (0)(4) - (-2)(-2) = 0 - 4 = \mathbf{-4}$

d. $\begin{vmatrix} 0 & 0 \\ 3 & 10 \end{vmatrix} = (0)(10) - (3)(0) = \mathbf{0}$

EXAMPLE D.2 Solve for x and y:

$$\begin{aligned} 2x + y &= 3 \\ 3x + 4y &= 2 \end{aligned}$$

Solution:

$$x = \frac{\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}} = \frac{(3)(4) - (2)(1)}{(2)(4) - (3)(1)} = \frac{12 - 2}{8 - 3} = \frac{10}{5} = \mathbf{2}$$

$$y = \frac{\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}}{5} = \frac{(2)(2) - (3)(3)}{5} = \frac{4 - 9}{5} = \frac{-5}{5} = \mathbf{-1}$$

Check:

$$
\begin{aligned}
2x + y &= (2)(2) + (-1) \\
&= 4 - 1 = 3 \qquad \text{(checks)} \\
3x + 4y &= (3)(2) + (4)(-1) \\
&= 6 - 4 = 2 \qquad \text{(checks)}
\end{aligned}
$$

EXAMPLE D.3 Solve for x and y:

$$
\begin{aligned}
-x + 2y &= 3 \\
\underline{3x - 2y} &\underline{= -2}
\end{aligned}
$$

Solution: In this example, note the effect of the minus sign and the use of parentheses to ensure that the proper sign is obtained for each product:

$$
x = \frac{\begin{vmatrix} 3 & 2 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix}} = \frac{(3)(-2) - (-2)(2)}{(-1)(-2) - (3)(2)}
$$

$$
= \frac{-6 + 4}{2 - 6} = \frac{-2}{-4} = \mathbf{\frac{1}{2}}
$$

$$
y = \frac{\begin{vmatrix} -1 & 3 \\ 3 & -2 \end{vmatrix}}{-4} = \frac{(-1)(-2) - (3)(3)}{-4}
$$

$$
= \frac{2 - 9}{-4} = \frac{-7}{-4} = \mathbf{\frac{7}{4}}
$$

EXAMPLE D.4 Solve for x and y:

$$
\begin{aligned}
x &= 3 - 4y \\
\underline{20y} &\underline{= -1 + 3x}
\end{aligned}
$$

Solution: In this case, the equations must first be placed in the format of Eqs. (D.1a) and (D.1b):

$$
\begin{aligned}
x + 4y &= 3 \\
\underline{-3x + 20y} &\underline{= -1}
\end{aligned}
$$

$$
x = \frac{\begin{vmatrix} 3 & 4 \\ -1 & 20 \end{vmatrix}}{\begin{vmatrix} 1 & 4 \\ -3 & 20 \end{vmatrix}} = \frac{(3)(20) - (-1)(4)}{(1)(20) - (-3)(4)}
$$

$$
= \frac{60 + 4}{20 + 12} = \frac{64}{32} = \mathbf{2}
$$

$$
y = \frac{\begin{vmatrix} 1 & 3 \\ -3 & -1 \end{vmatrix}}{32} = \frac{(1)(-1) - (-3)(3)}{32}
$$

$$
= \frac{-1 + 9}{32} = \frac{8}{32} = \mathbf{\frac{1}{4}}
$$

The use of determinants is not limited to the solution of two simultaneous equations; determinants can be applied to any number of simultaneous linear equations. First we examine a shorthand method that is applicable to third-order determinants only, since most of the problems in the text are limited to this level of difficulty. We then investigate the general procedure for solving any number of simultaneous equations.

Consider the three following simultaneous equations:

Col. 1		Col. 2		Col. 3		Col. 4
a_1x	$+$	b_1y	$+$	c_1z	$=$	d_1
a_2x	$+$	b_2y	$+$	c_2z	$=$	d_2
a_3x	$+$	b_3y	$+$	c_3z	$=$	d_3

in which x, y, and z are the variables, and $a_{1,2,3}$, $b_{1,2,3}$, $c_{1,2,3}$, and $d_{1,2,3}$ are constants.

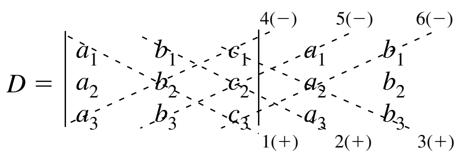
The determinant configuration for x, y, and z can be found in a manner similar to that for two simultaneous equations. That is, to solve for x, find the determinant in the numerator by replacing column 1 with the elements to the right of the equal sign. The denominator is the determinant of the coefficients of the variables (the same applies to y and z). Again, the denominator is the same for each variable.

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{D}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D}$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

A shorthand method for evaluating the third-order determinant consists of repeating the first two columns of the determinant to the right of the determinant and then summing the products along specific diagonals as shown below:



The products of the diagonals 1, 2, and 3 are positive and have the following magnitudes:

$$+a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3$$

The products of the diagonals 4, 5, and 6 are negative and have the following magnitudes:

$$-a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

The total solution is the sum of the diagonals 1, 2, and 3 minus the sum of the diagonals 4, 5, and 6:

$$+(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2c_1 + c_3a_2b_1) \qquad \textbf{(D.5)}$$

Warning: This method of expansion is good only for third-order determinants! It cannot be applied to fourth- and higher-order systems.

EXAMPLE D.5 Evaluate the following determinant:

$$\begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \\ 0 & 4 & 2 \end{vmatrix} \longrightarrow \begin{array}{ccc|cc} & & & (-) & (-) \quad (-) \\ 1 & 2 & 3 & 1 & 2 \\ -2 & 1 & 0 & -2 & 1 \\ 0 & 4 & 2 & 0 & 4 \\ & & & (+) & (+) \quad (+) \end{array}$$

Solution:

$$\begin{aligned} &[(1)(1)(2) + (2)(0)(0) + (3)(-2)(4)] \\ &\qquad - [(0)(1)(3) + (4)(0)(1) + (2)(-2)(2)] \\ &= (2 + 0 - 24) - (0 + 0 - 8) = (-22) - (-8) \\ &= -22 + 8 = \mathbf{-14} \end{aligned}$$

EXAMPLE D.6 Solve for *x*, *y*, and *z*:

$$\begin{aligned} 1x + 0y - 2z &= -1 \\ 0x + 3y + 1z &= +2 \\ 1x + 2y + 3z &= 0 \end{aligned}$$

Solution:

$$x = \frac{\begin{array}{ccc|cc} -1 & 0 & -2 & -1 & 0 \\ 2 & 3 & 1 & 2 & 3 \\ 0 & 2 & 3 & 0 & 2 \end{array}}{\begin{array}{ccc|cc} 1 & 0 & -2 & 1 & 0 \\ 0 & 3 & 1 & 0 & 3 \\ 1 & 2 & 3 & 1 & 2 \end{array}}$$

$$= \frac{[(-1)(3)(3) + (0)(1)(0) + (-2)(2)(2)] - [(0)(3)(-2) + (2)(1)(-1) + (3)(2)(0)]}{[(1)(3)(3) + (0)(1)(1) + (-2)(0)(2)] - [(1)(3)(-2) + (2)(1)(1) + (3)(0)(0)]}$$

$$= \frac{(-9 + 0 - 8) - (0 - 2 + 0)}{(9 + 0 + 0) - (-6 + 2 + 0)}$$

$$= \frac{-17 + 2}{9 + 4} = -\frac{\mathbf{15}}{\mathbf{13}}$$

$$y = \frac{\begin{array}{ccc|cc} 1 & -1 & -2 & 1 & -1 \\ 0 & 2 & 1 & 0 & 2 \\ 1 & 0 & 3 & 1 & 0 \end{array}}{13}$$

$$= \frac{[(1)(2)(3) + (-1)(1)(1) + (-2)(0)(0)] - [(1)(2)(-2) + (0)(1)(1) + (3)(0)(-1)]}{13}$$

$$= \frac{(6 - 1 + 0) - (-4 + 0 + 0)}{13}$$

$$= \frac{5 + 4}{13} = \frac{\mathbf{9}}{\mathbf{13}}$$

$$z = \frac{\left|\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 3 & 2 \\ 1 & 2 & 0 \end{array}\right|\begin{array}{cc} 1 & 0 \\ 0 & 3 \\ 1 & 2 \end{array}}{13}$$

$$= \frac{[(1)(3)(0) + (0)(2)(1) + (-1)(0)(2)] - [(1)(3)(-1) + (2)(2)(1) + (0)(0)(0)]}{13}$$

$$= \frac{(0 + 0 + 0) - (-3 + 4 + 0)}{13}$$

$$= \frac{0 - 1}{13} = \mathbf{-\frac{1}{13}}$$

or from $0x + 3y + 1z = +2$,

$$z = 2 - 3y = 2 - 3\left(\frac{9}{13}\right) = \frac{26}{13} - \frac{27}{13} = \mathbf{-\frac{1}{13}}$$

Check:

$$\left.\begin{array}{l} 1x + 0y - 2z = -1 \\[1em] 0x + 3y + 1z = +2 \\[1em] 1x + 2y + 3z = 0 \end{array}\right\} \left.\begin{array}{l} -\dfrac{15}{13} + 0 + \dfrac{2}{13} = -1 \\[1em] 0 + \dfrac{27}{13} + \dfrac{-1}{13} = +2 \\[1em] -\dfrac{15}{13} + \dfrac{18}{13} + \dfrac{-3}{13} = 0 \end{array}\right\} \begin{array}{l} -\dfrac{13}{13} = -1 \ \checkmark \\[1em] \dfrac{26}{13} = +2 \ \checkmark \\[1em] -\dfrac{18}{13} + \dfrac{18}{13} = 0 \ \checkmark \end{array}$$

The general approach to third-order or higher determinants requires that the determinant be expanded in the following form. There is more than one expansion that will generate the correct result, but this form is typically used when the material is first introduced.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \underset{\substack{\uparrow \\ \text{Multiplying factor}}}{a_1} \underbrace{\left(+\underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{Minor}}\right)}_{\text{Cofactor}} + \underset{\substack{\uparrow \\ \text{Multiplying factor}}}{b_1} \underbrace{\left(-\underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{Minor}}\right)}_{\text{Cofactor}} + \underset{\substack{\uparrow \\ \text{Multiplying factor}}}{c_1} \underbrace{\left(+\underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{Minor}}\right)}_{\text{Cofactor}}$$

This expansion was obtained by multiplying the elements of the first row of D by their corresponding cofactors. It is not a requirement that the first row be used as the multiplying factors. In fact, any *row* or *column* (not diagonals) may be used to expand a third-order determinant.

The sign of each cofactor is dictated by the position of the multiplying factors (a_1, b_1, and c_1 in this case) as in the following standard format:

$$\begin{vmatrix} + \rightarrow & - & + \\ \downarrow & & \\ - & + & - \\ & & \\ + & - & + \end{vmatrix}$$

Note that the proper sign for each element can be obtained by assigning the upper left element a positive sign and then changing signs as you move horizontally or vertically to the neighboring position.

For the determinant D, the elements would have the following signs:

$$\begin{vmatrix} a_1^{(+)} & b_1^{(-)} & c_1^{(+)} \\ a_2^{(-)} & b_2^{(+)} & c_2^{(-)} \\ a_3^{(+)} & b_3^{(-)} & c_3^{(+)} \end{vmatrix}$$

The minors associated with each multiplying factor are obtained by covering up the row and column in which the multiplying factor is located and writing a second-order determinant to include the remaining elements in the same relative positions that they have in the third-order determinant.

Consider the cofactors associated with a_1 and b_1 in the expansion of D. The sign is positive for a_1 and negative for b_1 as determined by the standard format. Following the procedure outlined above, we can find the minors of a_1 and b_1 as follows:

$$a_{1(\text{minor})} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$b_{1(\text{minor})} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

It was pointed out that any row or column may be used to expand the third-order determinant, and the same result will still be obtained. Using the first column of D, we obtain the expansion

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\left(+\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}\right) + a_2\left(-\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}\right) + a_3\left(+\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right)$$

The proper choice of row or column can often effectively reduce the amount of work required to expand the third-order determinant. For example, in the following determinants, the first column and third row, respectively, would reduce the number of cofactors in the expansion:

$$D = \begin{vmatrix} 2 & 3 & -2 \\ 0 & 4 & 5 \\ 0 & 6 & 7 \end{vmatrix} = 2\left(+\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix}\right) + 0 + 0 = 2(28 - 30)$$
$$= \mathbf{-4}$$

$$D = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 6 & 8 \\ 2 & 0 & 3 \end{vmatrix} = 2\left(+\begin{vmatrix} 4 & 7 \\ 6 & 8 \end{vmatrix}\right) + 0 + 3\left(+\begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix}\right)$$
$$= 2(32 - 42) + 3(6 - 8) = 2(-10) + 3(-2)$$
$$= \mathbf{-26}$$

EXAMPLE D.7 Expand the following third-order determinants:

a. $$D = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 1\left(+\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}\right) + 3\left(-\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}\right) + 2\left(+\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}\right)$$

$$= 1[6 - 1] + 3[-(6 - 3)] + 2[2 - 6]$$
$$= 5 + 3(-3) + 2(-4)$$
$$= 5 - 9 - 8$$
$$= \mathbf{-12}$$

b. $$D = \begin{vmatrix} 0 & 4 & 6 \\ 2 & 0 & 5 \\ 8 & 4 & 0 \end{vmatrix} = 0 + 2\left(-\begin{vmatrix} 4 & 6 \\ 4 & 0 \end{vmatrix}\right) + 8\left(+\begin{vmatrix} 4 & 6 \\ 0 & 5 \end{vmatrix}\right)$$

$$= 0 + 2[-(0 - 24)] + 8[(20 - 0)]$$
$$= 0 + 2(24) + 8(20)$$
$$= 48 + 160$$
$$= \mathbf{208}$$