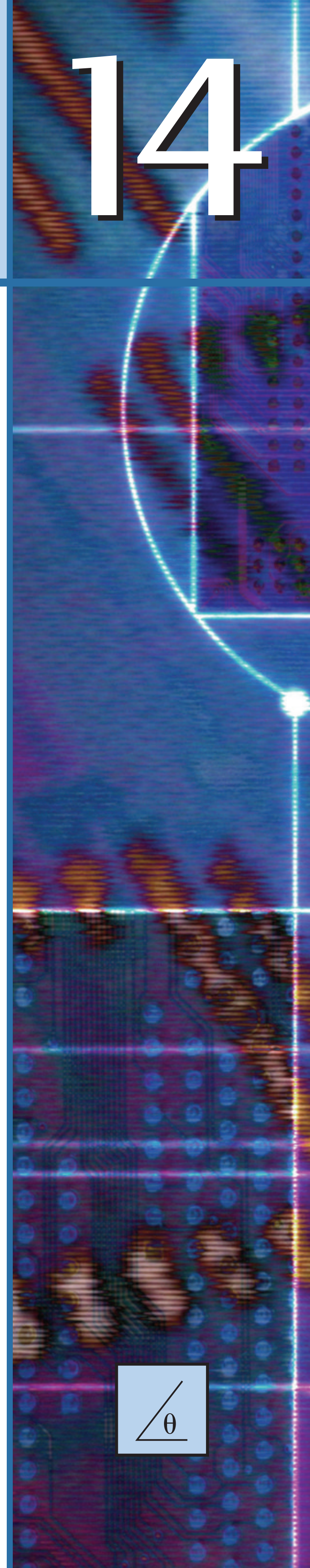

The Basic Elements and Phasors

Objectives

- *Become familiar with the response of a resistor, inductor, and capacitor to the application of a sinusoidal voltage or current.*
- *Learn how to apply the phasor format to add and subtract sinusoidal waveforms.*
- *Understand how to calculate the real power to resistive elements and the reactive power to inductive and capacitive elements.*
- *Become aware of the differences between the frequency response of ideal and practical elements.*
- *Become proficient in the use of a calculator or Mathcad to work with complex numbers.*

14.1 INTRODUCTION

The response of the basic *R, L,* and *C* elements to a sinusoidal voltage and current are examined in this chapter, with special note of how frequency affects the "opposing" characteristic of each element. Phasor notation is then introduced to establish a method of analysis that permits a direct correspondence with a number of the methods, theorems, and concepts introduced in the dc chapters.

14.2 DERIVATIVE

To understand the response of the basic *R, L,* and *C* elements to a sinusoidal signal, you need to examine the concept of the **derivative** in some detail. You do not have to become proficient in the mathematical technique but simply understand the impact of a relationship defined by a derivative.

Recall from Section 10.10 that the derivative dx/dt is defined as the rate of change of x with respect to time. If x fails to change at a particular instant, $dx = 0$, and the derivative is zero. For the sinusoidal waveform, dx/dt is zero only at the positive and negative peaks ($\omega t = \pi/2$ and $\frac{3}{2}\pi$ in Fig. 14.1), since x fails to change at these instants of time. The derivative dx/dt is actually the slope of the graph at any instant of time.

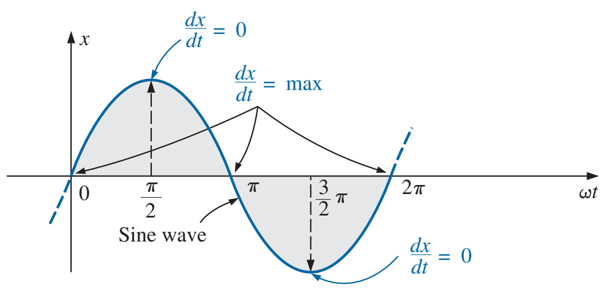

FIG. 14.1

Defining those points in a sinusoidal waveform that have maximum and minimum derivatives.

A close examination of the sinusoidal waveform will also indicate that the greatest change in x occurs at the instants $\omega t = 0$, π, and 2π. The derivative is therefore a maximum at these points. At 0 and 2π, x increases at its greatest rate, and the derivative is given a positive sign since x increases with time. At π, dx/dt decreases at the same rate as it increases at 0 and 2π, but the derivative is given a negative sign since x decreases with time. Since the rate of change at 0, π, and 2π is the same, the magnitude of the derivative at these points is the same also. For various values of ωt between these maxima and minima, the derivative will exist and have values from the minimum to the maximum inclusive. A plot of the derivative in Fig. 14.2 shows that

the derivative of a sine wave is a cosine wave.

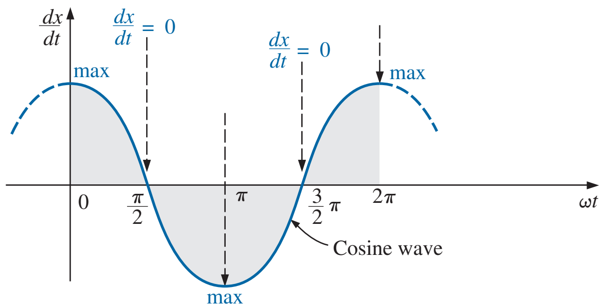

FIG. 14.2
Derivative of the sine wave of Fig. 14.1.

The peak value of the cosine wave is directly related to the frequency of the original waveform. The higher the frequency, the steeper the slope at the horizontal axis and the greater the value of dx/dt, as shown in Fig. 14.3 for two different frequencies.

Note in Fig. 14.3 that even though both waveforms (x_1 and x_2) have the same peak value, the sinusoidal function with the higher frequency produces the larger peak value for the derivative. In addition, note that

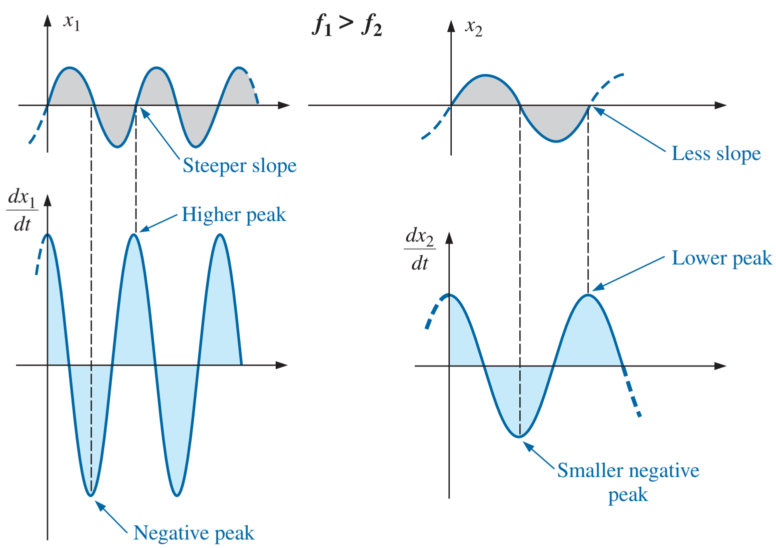

FIG. 14.3
Effect of frequency on the peak value of the derivative.

the derivative of a sine wave has the same period and frequency as the original sinusoidal waveform.

For the sinusoidal voltage

$$e(t) = E_m \sin(\omega t \pm \theta)$$

the derivative can be found directly by differentiation (calculus) to produce the following:

$$\begin{aligned}\frac{d}{dt}e(t) &= \omega E_m \cos(\omega t \pm \theta)\\ &= 2\pi f E_m \cos(\omega t \pm \theta)\end{aligned} \tag{14.1}$$

The mechanics of the differentiation process are not discussed or investigated here; nor are they required to continue with the text. Note, however, that the peak value of the derivative, $2\pi f E_m$, is a function of the frequency of $e(t)$, and the derivative of a sine wave is a cosine wave.

14.3 RESPONSE OF BASIC *R, L,* AND *C* ELEMENTS TO A SINUSOIDAL VOLTAGE OR CURRENT

Now that we are familiar with the characteristics of the derivative of a sinusoidal function, we can investigate the response of the basic elements *R, L,* and *C* to a sinusoidal voltage or current.

Resistor

For power-line frequencies and frequencies up to a few hundred kilohertz, resistance is, for all practical purposes, unaffected by the frequency of the applied sinusoidal voltage or current. For this frequency region, the resistor R in Fig. 14.4 can be treated as a constant, and Ohm's law can be applied as follows. For $\upsilon = V_m \sin \omega t$,

$$i = \frac{\upsilon}{R} = \frac{V_m \sin \omega t}{R} = \frac{V_m}{R}\sin \omega t = I_m \sin \omega t$$

where

$$I_m = \frac{V_m}{R} \tag{14.2}$$

In addition, for a given i,

$$\upsilon = iR = (I_m \sin \omega t)R = I_m R \sin \omega t = V_m \sin \omega t$$

where

$$V_m = I_m R \tag{14.3}$$

A plot of υ and i in Fig. 14.5 reveals that

for a purely resistive element, the voltage across and the current through the element are in phase, with their peak values related by Ohm's law.

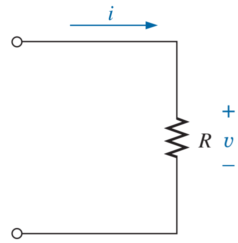

FIG. 14.4
Determining the sinusoidal response for a resistive element.

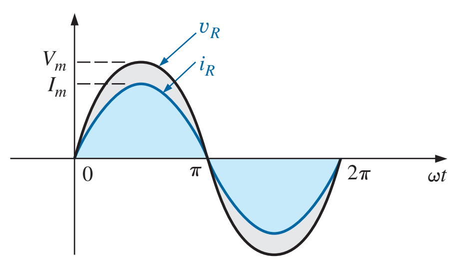

FIG. 14.5
The voltage and current of a resistive element are in phase.

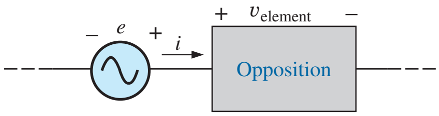

FIG. 14.6
Defining the opposition of an element to the flow of charge through the element.

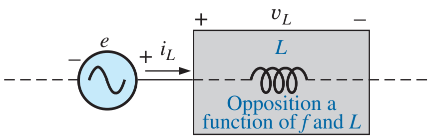

FIG. 14.7
Defining the parameters that determine the opposition of an inductive element to the flow of charge.

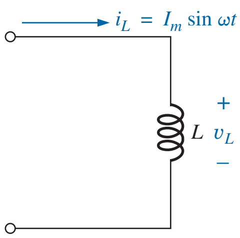

FIG. 14.8
Investigating the sinusoidal response of an inductive element.

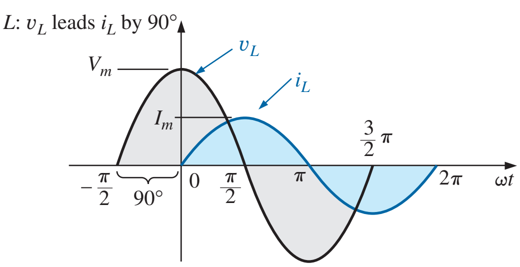

FIG. 14.9
For a pure inductor, the voltage across the coil leads the current through the coil by 90°.

Inductor

For the series configuration in Fig. 14.6, the voltage $v_{element}$ of the boxed-in element opposes the source e and thereby reduces the magnitude of the current i. The magnitude of the voltage across the element is determined by the opposition of the element to the flow of charge, or current i. For a resistive element, we have found that the opposition is its resistance and that $v_{element}$ and i are determined by $v_{element} = iR$.

We found in Chapter 11 that the voltage across an inductor is directly related to the rate of change of current through the coil. Consequently, the higher the frequency, the greater the rate of change of current through the coil, and the greater the magnitude of the voltage. In addition, we found in the same chapter that the inductance of a coil determines the rate of change of the flux linking a coil for a particular change in current through the coil. The higher the inductance, the greater the rate of change of the flux linkages, and the greater the resulting voltage across the coil.

The inductive voltage, therefore, is directly related to the frequency (or, more specifically, the angular velocity of the sinusoidal ac current through the coil) and the inductance of the coil. For increasing values of f and L in Fig. 14.7, the magnitude of v_L increases as described above.

Using the similarities between Figs. 14.6 and 14.7, we find that increasing levels of v_L are directly related to increasing levels of opposition in Fig. 14.6. Since v_L increases with both ω (= $2\pi f$) and L, the opposition of an inductive element is as defined in Fig. 14.7.

We will now verify some of the preceding conclusions using a more mathematical approach and then define a few important quantities to be used in the sections and chapters to follow.

For the inductor in Fig. 14.8, we recall from Chapter 11 that

$$v_L = L\frac{di_L}{dt}$$

and, applying differentiation,

$$\frac{di_L}{dt} = \frac{d}{dt}(I_m \sin \omega t) = \omega I_m \cos \omega t$$

Therefore, $$v_L = L\frac{di_L}{dt} = L(\omega I_m \cos \omega t) = \omega L I_m \cos \omega t$$

or $$v_L = V_m \sin(\omega t + 90°)$$

where $$V_m = \omega L I_m$$

Note that the peak value of v_L is directly related to ω (= $2\pi f$) and L as predicted in the discussion above.

A plot of v_L and i_L in Fig. 14.9 reveals that

for an inductor, v_L leads i_L by 90°, or i_L lags v_L by 90°.

If a phase angle is included in the sinusoidal expression for i_L, such as

$$i_L = I_m \sin(\omega t \pm \theta)$$

then $$v_L = \omega L I_m \sin(\omega t \pm \theta + 90°)$$

The opposition established by an inductor in a sinusoidal ac network can now be found by applying Eq. (4.1):

$$\text{Effect} = \frac{\text{cause}}{\text{opposition}}$$

which, for our purposes, can be written

$$\text{Opposition} = \frac{\text{cause}}{\text{effect}}$$

Substituting values, we have

$$\text{Opposition} = \frac{V_m}{I_m} = \frac{\omega L I_m}{I_m} = \omega L$$

revealing that the opposition established by an inductor in an ac sinusoidal network is directly related to the product of the angular velocity ($\omega = 2\pi f$) and the inductance, verifying our earlier conclusions.

The quantity ωL, called the **reactance** (from the word *reaction*) of an inductor, is symbolically represented by X_L and is measured in ohms; that is,

$$X_L = \omega L \qquad \text{(ohms, } \Omega) \qquad \textbf{(14.4)}$$

In an Ohm's law format, its magnitude can be determined from

$$X_L = \frac{V_m}{I_m} \qquad \text{(ohms, } \Omega) \qquad \textbf{(14.5)}$$

Inductive reactance is the opposition to the flow of current, which results in the continual interchange of energy between the source and the magnetic field of the inductor. In other words, inductive reactance, unlike resistance (which dissipates energy in the form of heat), does not dissipate electrical energy (ignoring the effects of the internal resistance of the inductor.)

Capacitor

Let us now return to the series configuration in Fig. 14.6 and insert the capacitor as the element of interest. For the capacitor, however, we will determine i for a particular voltage across the element. When this approach reaches its conclusion, we will know the relationship between the voltage and current and can determine the opposing voltage (v_{element}) for any sinusoidal current i.

Our investigation of the inductor revealed that the inductive voltage across a coil opposes the instantaneous change in current through the coil. For capacitive networks, the voltage across the capacitor is limited by the rate at which charge can be deposited on, or released by, the plates of the capacitor during the charging and discharging phases, respectively. In other words, an instantaneous change in voltage across a capacitor is opposed by the fact that there is an element of time required to deposit charge on (or release charge from) the plates of a capacitor, and $V = Q/C$.

Since capacitance is a measure of the rate at which a capacitor will store charge on its plates,

for a particular change in voltage across the capacitor, the greater the value of capacitance, the greater the resulting capacitive current.

In addition, the fundamental equation relating the voltage across a capacitor to the current of a capacitor [$i = C(dv/dt)$] indicates that

for a particular capacitance, the greater the rate of change of voltage across the capacitor, the greater the capacitive current.

Certainly, an increase in frequency corresponds to an increase in the rate of change of voltage across the capacitor and to an increase in the current of the capacitor.

The current of a capacitor is therefore directly related to the frequency (or, again more specifically, the angular velocity) and the capacitance of the capacitor. An increase in either quantity results in an increase in the current of the capacitor. For the basic configuration in Fig. 14.10, however, we are interested in determining the opposition of the capacitor as related to the resistance of a resistor and ωL for the inductor. Since an increase in current corresponds to a decrease in opposition, and i_C is proportional to ω and C, the opposition of a capacitor is inversely related to ω $(= 2\pi f)$ and C.

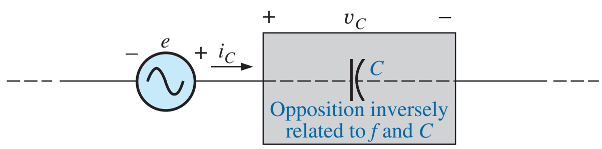

FIG. 14.10
Defining the parameters that determine the opposition of a capacitive element to the flow of charge.

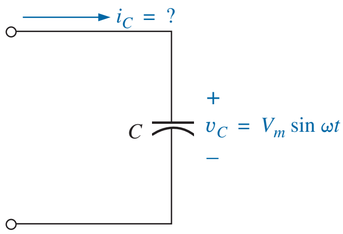

FIG. 14.11
Investigating the sinusoidal response of a capacitive element.

We will now verify, as we did for the inductor, some of the above conclusions using a more mathematical approach.

For the capacitor of Fig. 14.11, we recall from Chapter 11 that

$$i_C = C\frac{dv_C}{dt}$$

and, applying differentiation,

$$\frac{dv_C}{dt} = \frac{d}{dt}(V_m \sin \omega t) = \omega V_m \cos \omega t$$

Therefore,

$$i_C = C\frac{dv_C}{dt} = C(\omega V_m \cos \omega t) = \omega C V_m \cos \omega t$$

or $$i_C = I_m \sin(\omega t + 90°)$$

where $$I_m = \omega C V_m$$

Note that the peak value of i_C is directly related to ω $(= 2\pi f)$ and C, as predicted in the discussion above.

A plot of v_C and i_C in Fig. 14.12 reveals that

for a capacitor, i_C leads v_C by 90°, or v_C lags i_C by 90°.*

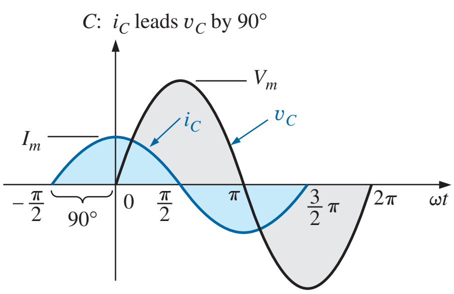

FIG. 14.12
The current of a purely capacitive element leads the voltage across the element by 90°.

If a phase angle is included in the sinusoidal expression for v_C, such as

$$v_C = V_m \sin(\omega t \pm \theta)$$

then $$i_C = \omega C V_m \sin(\omega t \pm \theta + 90°)$$

*A mnemonic phrase sometimes used to remember the phase relationship between the voltage and current of a coil and capacitor is "*ELI* the *ICE* man." Note that the *L* (inductor) has the *E* before the *I* (*e* leads *i* by 90°), and the *C* (capacitor) has the *I* before the *E* (*i* leads *e* by 90°).

Applying

$$\text{Opposition} = \frac{\text{cause}}{\text{effect}}$$

and substituting values, we obtain

$$\text{Opposition} = \frac{V_m}{I_m} = \frac{V_m}{\omega C V_m} = \frac{1}{\omega C}$$

which agrees with the results obtained above.

The quantity $1/\omega C$, called the **reactance** of a capacitor, is symbolically represented by X_C and is measured in ohms; that is,

$$\boxed{X_C = \frac{1}{\omega C}} \qquad \text{(ohms, } \Omega) \qquad \textbf{(14.6)}$$

In an Ohm's law format, its magnitude can be determined from

$$\boxed{X_C = \frac{V_m}{I_m}} \qquad \text{(ohms, } \Omega) \qquad \textbf{(14.7)}$$

Capacitive reactance is the opposition to the flow of charge, which results in the continual interchange of energy between the source and the electric field of the capacitor. Like the inductor, the capacitor does *not* dissipate energy in any form (ignoring the effects of the leakage resistance).

In the circuits just considered, the current was given in the inductive circuit, and the voltage in the capacitive circuit. This was done to avoid the use of integration in finding the unknown quantities. In the inductive circuit,

$$\upsilon_L = L\frac{di_L}{dt}$$

but

$$\boxed{i_L = \frac{1}{L}\int \upsilon_L\, dt} \qquad \textbf{(14.8)}$$

In the capacitive circuit,

$$i_C = C\frac{d\upsilon_C}{dt}$$

but

$$\boxed{\upsilon_C = \frac{1}{C}\int i_C\, dt} \qquad \textbf{(14.9)}$$

Shortly, we shall consider a method of analyzing ac circuits that will permit us to solve for an unknown quantity with sinusoidal input without having to use direct integration or differentiation.

It is possible to determine whether a network with one or more elements is predominantly capacitive or inductive by noting the phase relationship between the input voltage and current.

If the source current leads the applied voltage, the network is predominantly capacitive, and if the applied voltage leads the source current, it is predominantly inductive.

Since we now have an equation for the reactance of an inductor or capacitor, we do not need to use derivatives or integration in the examples

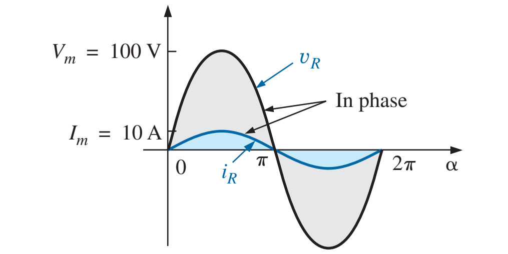

FIG. 14.13
Example 14.1(a).

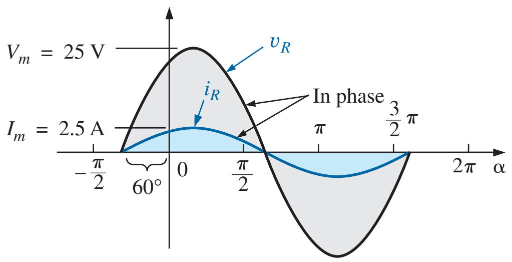

FIG. 14.14
Example 14.1(b).

to be considered. Simply applying Ohm's law, $I_m = E_m/X_L$ (or X_C), and keeping in mind the phase relationship between the voltage and current for each element, will be sufficient to complete the examples.

EXAMPLE 14.1 The voltage across a resistor is indicated. Find the sinusoidal expression for the current if the resistor is 10 Ω. Sketch the curves for v and i.

a. $v = 100 \sin 377t$
b. $v = 25 \sin(377t + 60°)$

Solutions:

a. Eq. (14.2): $I_m = \frac{V_m}{R} = \frac{100\text{ V}}{10\ \Omega} = 10\text{ A}$

(v and i are in phase), resulting in

$$i = \mathbf{10 \sin 377}\boldsymbol{t}$$

The curves are sketched in Fig. 14.13.

b. Eq. (14.2): $I_m = \frac{V_m}{R} = \frac{25\text{ V}}{10\ \Omega} = 2.5\text{ A}$

(v and i are in phase), resulting in

$$i = \mathbf{2.5 \sin(377}\boldsymbol{t}\mathbf{ + 60°)}$$

The curves are sketched in Fig. 14.14.

EXAMPLE 14.2 The current through a 5 Ω resistor is given. Find the sinusoidal expression for the voltage across the resistor for $i = 40 \sin(377t + 30°)$.

Solution: Eq. (14.3): $V_m = I_mR = (40\text{ A})(5\ \Omega) = 200\text{ V}$

(v and i are in phase), resulting in

$$v = \mathbf{200 \sin(377}\boldsymbol{t}\mathbf{ + 30°)}$$

EXAMPLE 14.3 The current through a 0.1 H coil is provided. Find the sinusoidal expression for the voltage across the coil. Sketch the v and i curves.

a. $i = 10 \sin 377t$
b. $i = 7 \sin(377t - 70°)$

Solutions:

a. Eq. (14.4): $X_L = \omega L = (377\text{ rad/s})(0.1\text{ H}) = 37.7\ \Omega$
Eq. (14.5): $V_m = I_mX_L = (10\text{ A})(37.7\ \Omega) = 377\text{ V}$

and we know that for a coil v leads i by 90°. Therefore,

$$v = \mathbf{377 \sin(377}\boldsymbol{t}\mathbf{ + 90°)}$$

The curves are sketched in Fig. 14.15.

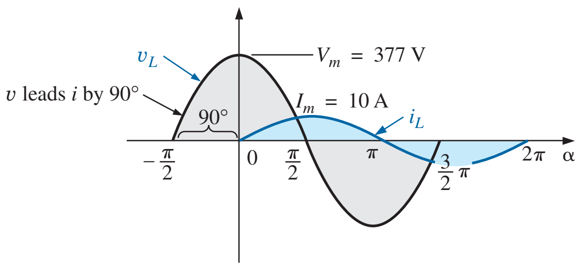

FIG. 14.15
Example 14.3(a).

b. X_L remains at 37.7 Ω.

$V_m = I_m X_L = (7\text{ A})(37.7\ \Omega) = 263.9\text{ V}$

and we know that for a coil υ leads i by 90°. Therefore,

$$\upsilon = 263.9 \sin(377t - 70° + 90°)$$

and

$$\boldsymbol{\upsilon = 263.9 \sin(377t + 20°)}$$

The curves are sketched in Fig. 14.16.

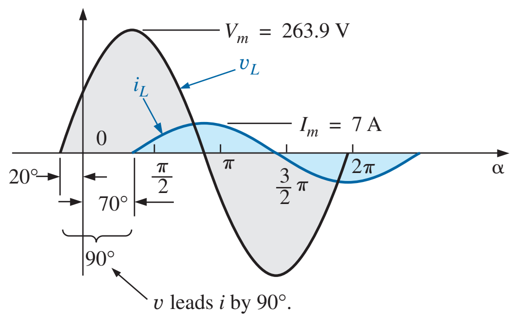

FIG. 14.16
Example 14.3(b).

EXAMPLE 14.4 The voltage across a 0.5 H coil is provided below. What is the sinusoidal expression for the current?

$$\upsilon = 100 \sin 20t$$

Solution:

$$X_L = \omega L = (20\text{ rad/s})(0.5\text{ H}) = 10\ \Omega$$

$$I_m = \frac{V_m}{X_L} = \frac{100\text{ V}}{10\ \Omega} = 10\text{ A}$$

and we know the i lags υ by 90°. Therefore,

$$i = \mathbf{10 \sin(20}\boldsymbol{t}\mathbf{ - 90°)}$$

EXAMPLE 14.5 The voltage across a 1 μF capacitor is provided below. What is the sinusoidal expression for the current? Sketch the υ and i curves.

$$\upsilon = 30 \sin 400t$$

Solution:

Eq. (14.6): $$X_C = \frac{1}{\omega C} = \frac{1}{(400\text{ rad/s})(1 \times 10^{-6}\text{ F})} = \frac{10^6\ \Omega}{400} = 2500\ \Omega$$

Eq. (14.7): $$I_m = \frac{V_m}{X_C} = \frac{30\text{ V}}{2500\ \Omega} = 0.0120\text{ A} = 12\text{ mA}$$

and we know that for a capacitor i leads υ by 90°. Therefore,

$$i = \mathbf{12 \times 10^{-3} \sin(400}\boldsymbol{t}\mathbf{ + 90°)}$$

The curves are sketched in Fig. 14.17.

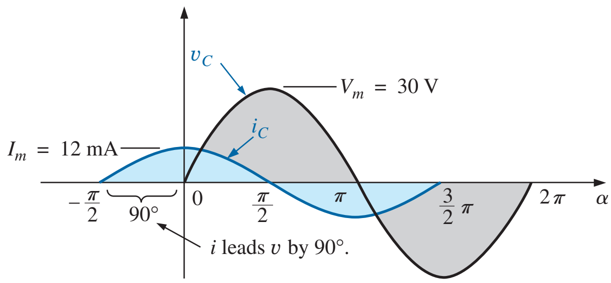

FIG. 14.17
Example 14.5.

EXAMPLE 14.6 The current through a 100 μF capacitor is given. Find the sinusoidal expression for the voltage across the capacitor.

$$i = 40 \sin(500t + 60°)$$

Solution:

$$X_C = \frac{1}{\omega C} = \frac{1}{(500 \text{ rad/s})(100 \times 10^{-6} \text{ F})} = \frac{10^6 \ \Omega}{5 \times 10^4} = \frac{10^2 \ \Omega}{5} = 20 \ \Omega$$

$$V_M = I_M X_C = (40 \text{ A})(20 \ \Omega) = 800 \text{ V}$$

and we know that for a capacitor, v lags i by 90°. Therefore,

$$v = 800 \sin(500t + 60° - 90°)$$

and $$\mathbf{v = 800 \sin(500t - 30°)}$$

EXAMPLE 14.7 For the following pairs of voltages and currents, determine whether the element involved is a capacitor, an inductor, or a resistor. Determine the value of C, L, or R if sufficient data are provided (Fig. 14.18):

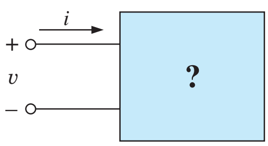

FIG. 14.18
Example 14.7.

a. $v = 100 \sin(\omega t + 40°)$
   $i = 20 \sin(\omega t + 40°)$
b. $v = 1000 \sin(377t + 10°)$
   $i = 5 \sin(377t - 80°)$
c. $v = 500 \sin(157t + 30°)$
   $i = 1 \sin(157t + 120°)$
d. $v = 50 \cos(\omega t + 20°)$
   $i = 5 \sin(\omega t + 110°)$

Solutions:

a. Since v and i are *in phase*, the element is a *resistor*, and

$$R = \frac{V_m}{I_m} = \frac{100 \text{ V}}{20 \text{ A}} = \mathbf{5 \ \Omega}$$

b. Since v *leads* i by 90°, the element is an *inductor*, and

$$X_L = \frac{V_m}{I_m} = \frac{1000 \text{ V}}{5 \text{ A}} = 200 \ \Omega$$

so that $X_L = \omega L = 200 \ \Omega$ or

$$L = \frac{200\ \Omega}{\omega} = \frac{200\ \Omega}{377\ \text{rad/s}} = \mathbf{0.53\ H}$$

c. Since i *leads* v by 90°, the element is a *capacitor,* and

$$X_C = \frac{V_m}{I_m} = \frac{500\ \text{V}}{1\ \text{A}} = 500\ \Omega$$

so that $X_C = \dfrac{1}{\omega C} = 500\ \Omega$ or

$$C = \frac{1}{\omega 500\ \Omega} = \frac{1}{(157\ \text{rad/s})(500\ \Omega)} = \mathbf{12.74\ \mu F}$$

d. $v = 50\cos(\omega t + 20°) = 50\sin(\omega t + 20° + 90°)$
$= 50\sin(\omega t + 110°)$

Since v and i are *in phase,* the element is a *resistor,* and

$$R = \frac{V_m}{I_m} = \frac{50\ \text{V}}{5\ \text{A}} = \mathbf{10\ \Omega}$$

14.4 FREQUENCY RESPONSE OF THE BASIC ELEMENTS

Thus far, each description has been for a set frequency, resulting in a fixed level of impedance for each of the basic elements. We must now investigate how a change in frequency affects the impedance level of the basic elements. It is an important consideration because most signals other than those provided by a power plant contain a variety of frequency levels. The last section made it quite clear that the reactance of an inductor or a capacitor is sensitive to the applied frequency. However, the question is, How will these reactance levels change if we steadily increase the frequency from a very low level to a much higher level?

Although we would like to think of every element as ideal, it is important to realize that every commercial element available today *will not respond in an ideal fashion for the full range of possible frequencies.* That is, each element is such that for a particular range of frequencies, it performs in an essentially ideal manner. However, there is always a range of frequencies in which the performance varies from the ideal. Fortunately, the designer is aware of these limitations and will take them into account in the design.

The discussion begins with a look at the response of the *ideal elements*—a response that will be assumed for the remaining chapters of this text and one that can be assumed for any initial investigation of a network. This discussion is followed by a look at the factors that cause an element to deviate from an ideal response as frequency levels become too low or high.

Ideal Response

Resistor *R* For an ideal resistor, you can assume that *frequency will have absolutely no effect on the impedance level,* as shown by the response in Fig. 14.19. Note that at 5 kHz or 20 kHz, the resistance of the resistor remains at 22 Ω; there is no change whatsoever. For the rest of

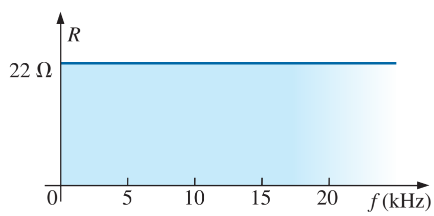

FIG. 14.19
R versus f for the range of interest.

the analyses in this text, the resistance level remains as the nameplate value, no matter what frequency is applied.

Inductor *L* For the ideal inductor, the equation for the reactance can be written as follows to isolate the frequency term in the equation. The result is a constant times the frequency variable that changes as we move down the horizontal axis of a plot:

$$X_L = \omega L = 2\pi f L = (2\pi L)f = kf \quad \text{with } k = 2\pi L$$

The resulting equation can be compared directly with the equation for a straight line:

$$y = mx + b = kf + 0 = kf$$

where $b = 0$ and the slope is k or $2\pi L$. X_L is the y variable, and f is the x variable, as shown in Fig. 14.20. Since the inductance determines the slope of the curve, the higher the inductance, the steeper the straight-line plot as shown in Fig. 14.20 for two levels of inductance.

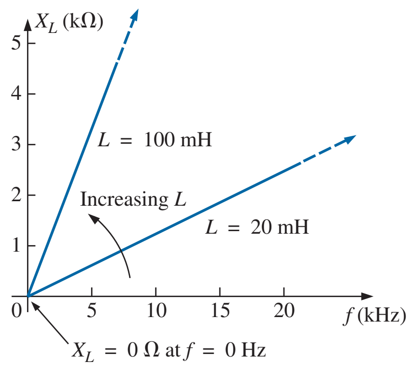

FIG. 14.20
X_L versus frequency.

In particular, note that at $f = 0$ Hz, the reactance of each plot is zero ohms as determined by substituting $f = 0$ Hz into the basic equation for the reactance of an inductor:

$$X_L = 2\pi f L = 2\pi(0 \text{ Hz})L = 0\ \Omega$$

Since a reactance of zero ohms corresponds with the characteristics of a short circuit, we can conclude that

at a frequency of 0 Hz, an inductor takes on the characteristics of a short circuit, as shown in Fig. 14.21.

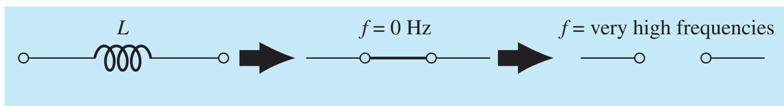

FIG. 14.21
Effect of low and high frequencies on the circuit model of an inductor.

As shown in Fig. 14.21, as the frequency increases, the reactance increases, until it reaches an extremely high level at very high frequencies. The result is that

at very high frequencies, the characteristics of an inductor approach those of an open circuit, as shown in Fig. 14.21.

The inductor, therefore, is capable of handling impedance levels that cover the entire range, from zero ohms to infinite ohms, changing at a *steady rate* determined by the inductance level. The higher the inductance, the faster it approaches the open-circuit equivalent.

Capacitor *C* For the capacitor, the equation for the reactance

$$X_C = \frac{1}{2\pi f C}$$

can be written as

$$X_C f = \frac{1}{2\pi C} = k \qquad \text{(a constant)}$$

which matches the basic format for a hyberbola:

$$yx = k$$

where X_C is the y variable, f the x variable, and k a constant equal to $1/(2\pi C)$.

Hyberbolas have the shape appearing in Fig. 14.22 for two levels of capacitance. Note that the higher the capacitance, the closer the curve approaches the vertical and horizontal axes at low and high frequencies.

At or near 0 Hz, the reactance of any capacitor is extremely high, as determined by the basic equation for capacitance:

$$X_C = \frac{1}{2\pi fc} = \frac{1}{2\pi(0\text{ Hz})C} \Rightarrow \infty\ \Omega$$

The result is that

at or near 0 Hz, the characteristics of a capacitor approach those of an open circuit, as shown in Fig. 14.23.

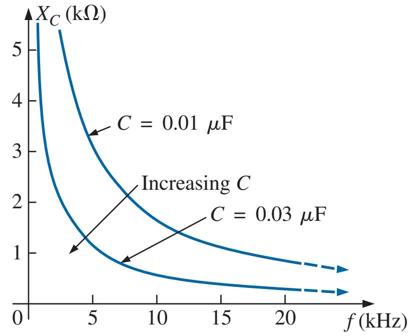

FIG. 14.22
X_C versus frequency.

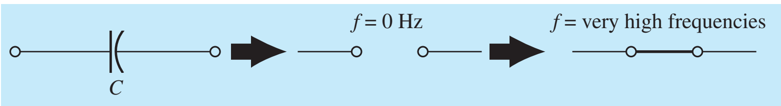

FIG. 14.23
Effect of low and high frequencies on the circuit model of a capacitor.

As the frequency increases, the reactance approaches a value of zero ohms. The result is that

at very high frequencies, a capacitor takes on the characteristics of a short circuit, as shown in Fig. 14.23.

It is important to note in Fig. 14.22 that the reactance drops very rapidly as the frequency increases. It is not a gradual drop as encountered for the rise in inductive reactance. In addition, the reactance sits at a fairly low level for a broad range of frequencies. In general, therefore, recognize that for capacitive elements, the change in reactance level can be dramatic with a relatively small change in frequency level.

Finally, recognize the following:

As frequency increases, the reactance of an inductive element increases while that of a capacitor decreases, with one approaching an open-circuit equivalent as the other approaches a short-circuit equivalent.

Practical Response

Resistor *R* In the manufacturing process, every resistive element inherits some stray capacitance levels and lead inductances. For most applications, the levels are so low that their effects can be ignored. However, as the frequency extends beyond a few megahertz, it may be necessary to be aware of their effects. For instance, a number of carbon composition resistors have the frequency response appearing in Fig. 14.24. The 100 Ω resistor is essentially stable up to about 300 MHz, whereas the 100 kΩ resistor starts to drop off at about 15 MHz. In general, therefore, this type of carbon composition resistor has the ideal characteristics of Fig. 14.19 for frequencies up to about 15 MHz. For frequencies of 100 Hz, 1 kHz, 150 kHz, and so on, the resistor can be considered ideal.

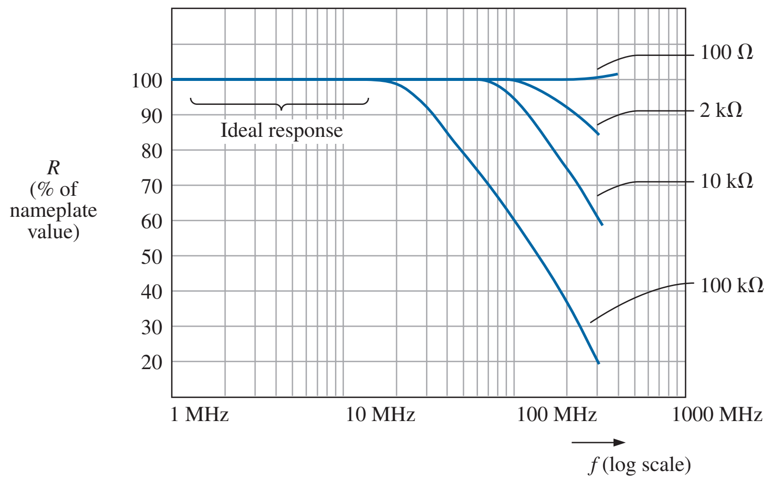

FIG. 14.24
Typical resistance-versus-frequency curves for carbon composition resistors.

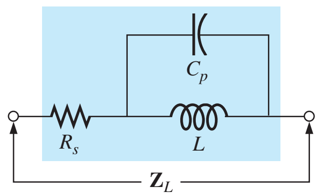

FIG. 14.25
Practical equivalent for an inductor.

Inductor *L* In reality, inductance can be affected by frequency, temperature, and current. A true equivalent for an inductor appears in Fig. 14.25. The series resistance R_s represents the copper losses (resistance of the many turns of thin copper wire); the eddy current losses (losses due to small circular currents in the core when an ac voltage is applied); and the hysteresis losses (losses due to core losses created by the rapidly reversing field in the core). The capacitance C_p is the stray capacitance that exists between the windings of the inductor.

For most inductors, the construction is usually such that the larger the inductance, the lower the frequency at which the parasitic elements become important. That is, for inductors in the millihenry range (which is very typical), frequencies approaching 100 kHz can have an effect on the ideal characteristics of the element. For inductors in the microhenry range, a frequency of 1 MHz may introduce negative effects. This is not to suggest that the inductors lose their effect at these frequencies but rather that they can no longer be considered ideal (purely inductive elements).

Fig. 14.26 is a plot of the magnitude of the impedance Z_L of Fig. 14.25 versus frequency. Note that up to about 2 MHz, the impedance increases almost linearly with frequency, clearly suggesting that the 100 μH inductor is essentially ideal. However, above 2 MHz, all the factors contributing to R_s start to increase, while the reactance due to the capacitive element C_p

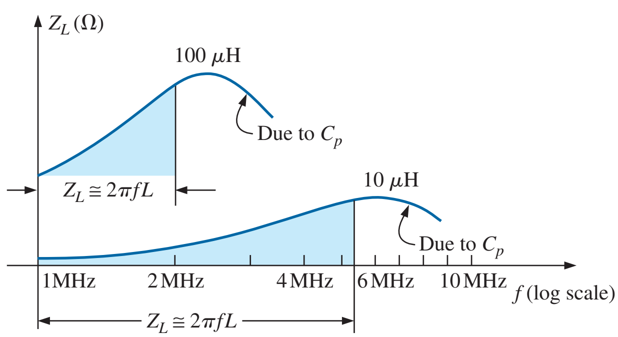

FIG. 14.26
Z_L versus frequency for the practical inductor equivalent of Fig. 14.25.

is more pronounced. The dropping level of capacitive reactance begins to have a shorting effect across the windings of the inductor and reduces the overall inductive effect. Eventually, if the frequency continues to increase, the capacitive effects overcome the inductive effects, and the element actually begins to behave in a capacitive fashion. Note the similarities of this region with the curves in Fig. 14.22. Also, note that decreasing levels of inductance (available with fewer turns and therefore lower levels of C_p) do not demonstrate the degrading effect until higher frequencies are applied.

In general, therefore, the frequency of application for a coil becomes important at increasing frequencies. Inductors lose their ideal characteristics and, in fact, begin to act as capacitive elements with increasing losses at very high frequencies.

Capacitor *C* The capacitor, like the inductor, is not ideal at higher frequencies. In fact, a transition point can be defined where the characteristics of the capacitor will actually be inductive. The complete equivalent model for a capacitor is provided in Fig. 14.27. The resistance R_s, defined by the resistivity of the dielectric (typically $10^{12}\ \Omega \cdot$ m or better) and the case resistance, determines the level of leakage current to expect during the discharge cycle. In other words, a charged capacitor can discharge both through the case and through the dielectric at a rate determined by the resistance of each path. Depending on the capacitor, the discharge time can extend from a few seconds for some electrolytic capacitors to hours (paper) or perhaps days (polystyrene). Inversely, therefore, electrolytics obviously have much lower levels of R_s than paper or polystyrene.

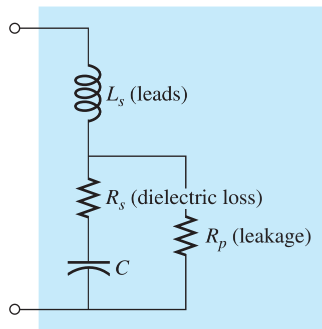

FIG. 14.27
Practical equivalent for a capacitor.

The resistance R_p reflects the energy lost as the atoms continually realign themselves in the dielectric due to the applied alternating ac voltage. Molecular friction is present due to the motion of the atoms as they respond to the alternating applied electric field. Interestingly enough, however, the relative permittivity decreases with increasing frequencies but eventually takes a complete turnaround and begins to increase at very high frequencies. The inductance L_s includes the inductance of the capacitor leads and any inductive effects introduced by the design of the capacitor. Be aware that the inductance of the leads is about 0.05 μH per centimeter or 0.2 μH for a capacitor with two 2 cm leads—a level that can be important at high frequencies. As for the inductor, the capacitor behaves quite ideally for the low- and mid-frequency range, as shown by the plot in Fig. 14.28 for a 0.01 μF metalized film capacitor with 2 cm

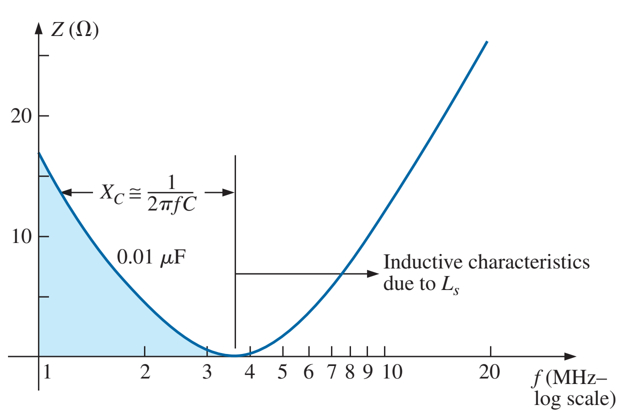

FIG. 14.28
Impedance characteristics of a 0.01 μF metalized film capacitor versus frequency.

leads. As the frequency increases, however, and the reactance X_s becomes larger, a frequency is eventually reached where the reactance of the coil equals that of the capacitor (a resonant condition to be described in Chapter 20). Any additional increase in frequency results in X_s being greater than X_C, and the element behaves like an inductor.

In general, therefore, the frequency of application is important for capacitive elements because when the frequency increases to a certain level, the element takes on inductive characteristics. Also, the frequency of application defines the type of capacitor (or inductor) that is applied: Electrolytics are limited to frequencies to perhaps 10 kHz, while ceramic or mica can handle frequencies higher than 10 MHz.

The expected temperature range of operation can have an important impact on the type of capacitor chosen for a particular application. Electrolytics, tantalum, and some high-*k* ceramic capacitors are very sensitive to colder temperatures. In fact, most electrolytics lose 20% of their room-temperature capacitance at 0°C (freezing). Higher temperatures (up to 100°C or 212°F) seem to have less impact in general than colder temperatures, but high-*k* ceramics can lose up to 30% of their capacitance level at 100°C compared to room temperature. With experience, you will learn the type of capacitor to use for each application and only be concerned when you encounter very high frequencies, extreme temperatures, or very high currents or voltages.

EXAMPLE 14.8 At what frequency will the reactance of a 200 mH inductor match the resistance level of a 5 kΩ resistor?

Solution: The resistance remains constant at 5 kΩ for the frequency range of the inductor. Therefore,

$$R = 5000\ \Omega = X_L = 2\pi f L = 2\pi L f$$
$$= 2\pi(200 \times 10^{-3}\ \text{H})f = 1.257f$$

and

$$f = \frac{5000\ \text{Hz}}{1.257} \cong \mathbf{3.98\ kHz}$$

EXAMPLE 14.9 At what frequency will an inductor of 5 mH have the same reactance as a capacitor of 0.1 μF?

Solution:

$$X_L = X_C$$
$$2\pi f L = \frac{1}{2\pi f C}$$
$$f^2 = \frac{1}{4\pi^2 LC}$$

and

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(5 \times 10^{-3}\ \text{H})(0.1 \times 10^{-6}\ \text{F})}}$$
$$= \frac{1}{2\pi\sqrt{5 \times 10^{-10}}} = \frac{1}{(2\pi)(2.236 \times 10^{-5})} = \frac{10^5\ \text{Hz}}{14.05} \cong \mathbf{7.12\ kHz}$$

14.5 AVERAGE POWER AND POWER FACTOR

A common question is, How can a sinusoidal voltage or current deliver power to a load if it seems to be delivering power during one part of its cycle and taking it back during the negative part of the sinusoidal cycle? The equal oscillations above and below the axis seem to suggest that over one full cycle there is no net transfer of power or energy. However, as mentioned in the last chapter, there is a net transfer of power over one full cycle because power is delivered to the load *at each instant* of the applied voltage or current (except when either is crossing the axis) no matter what the direction is of the current or polarity of the voltage.

To demonstrate this, consider the relatively simple configuration in Fig. 14.29 where an 8 V peak sinusoidal voltage is applied across a 2 Ω resistor. When the voltage is at its positive peak, the power delivered at that instant is 32 W as shown in the figure. At the midpoint of 4 V, the instantaneous power delivered drops to 8 W; when the voltage crosses the axis, it drops to 0 W. Note, however, that when the applied voltage is at its negative peak, the current may reverse but, at that instant, 32 W is still being delivered to the resistor.

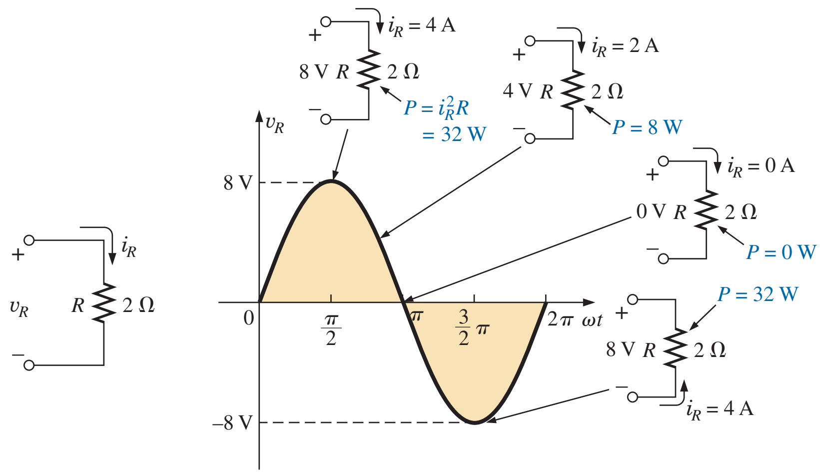

FIG. 14.29
Demonstrating that power is delivered at every instant of a sinusoidal voltage waveform (except $v_R = 0$ V).

In total, therefore,

even though the current through and the voltage across reverse direction and polarity, respectively, power is delivered to the resistive load at each instant of time.

If we plot the power delivered over a full cycle, the curve in Fig. 14.30 results. Note that the applied voltage and resulting current are in phase and have twice the frequency of the power curve. For one full cycle of the applied voltage having a period *T*, the power level peaks for each pulse of the sinusoidal waveform.

The fact that the power curve is always above the horizontal axis reveals that power is being delivered to the load at each instant of time of the applied sinusoidal voltage.

Any portion of the power curve below the axis reveals that power is being returned to the source. The average value of the power curve occurs at a level equal to $V_m I_m/2$ as shown in Fig. 14.30. This power level

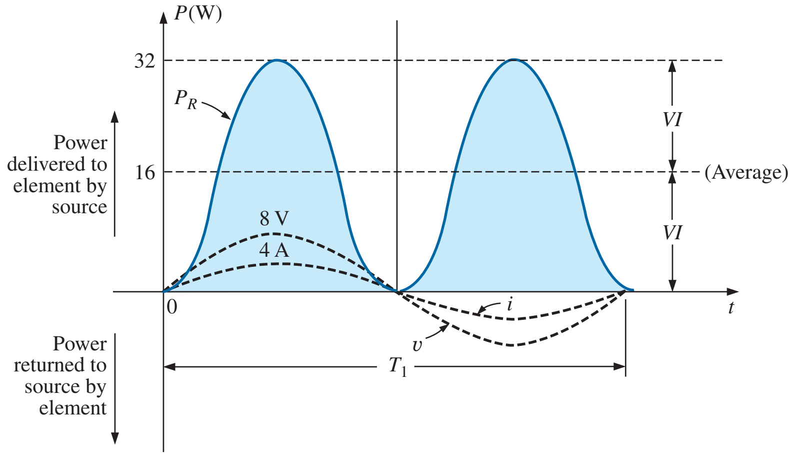

FIG. 14.30
Power versus time for a purely resistive load.

is called the **average** or **real power** level. It establishes a particular level of power transfer for the full cycle, so that we do not have to determine the level of power to apply to a quantity that varies in a sinusoidal nature.

If we substitute the equation for the peak value in terms of the rms value as follows:

$$P_{av} = \frac{V_m I_m}{2} = \frac{(\sqrt{2}\,V_{rms})(\sqrt{2}\,I_{rms})}{2} = \frac{2\,V_{rms}I_{rms}}{2}$$

we find that the average or real power delivered to a resistor takes on the following very convenient form:

$$P_{av} = V_{rms}\,I_{rms} \tag{14.10}$$

Note that the power equation is exactly the same when applied to dc networks as long as we work with rms values.

The above analysis was for a purely resistive load. If the sinusoidal voltage is applied to a network with a combination of *R, L,* and *C* components, the instantaneous equation for the power levels is more complex. However, if we are careful in developing the general equation and examine the results, we find some general conclusions that will be very helpful in the analysis to follow.

In Fig. 14.31, a voltage with an initial phase angle is applied to a network with any combination of elements that results in a current with the indicated phase angle.

The power delivered at each instant of time is then defined by

$$\begin{aligned} p = vi &= V_m \sin(\omega t + \theta_v) I_m \sin(\omega t + \theta_i) \\ &= V_m I_m \sin(\omega t + \theta_v)\sin(\omega t + \theta_i) \end{aligned}$$

Using the trigonometric identity

$$\sin A \sin B = \frac{\cos(A - B) - \cos(A + B)}{2}$$

the function $\sin(\omega t + \theta_v)\sin(\omega t + \theta_i)$ becomes

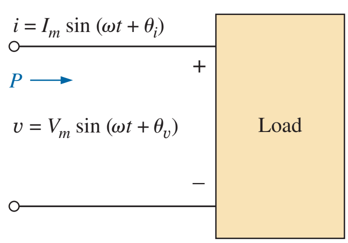

FIG. 14.31
Determining the power delivered in a sinusoidal ac network.

$$\sin(\omega t + \theta_v)\sin(\omega t + \theta_i) = \frac{\cos[(\omega t + \theta_v) - (\omega t + \theta_i)] - \cos[(\omega t + \theta_v) + (\omega t + \theta_i)]}{2} = \frac{\cos(\theta_v - \theta_i) - \cos(2\omega t + \theta_v + \theta_i)}{2}$$

so that

$$p = \overbrace{\left[\frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\right]}^{\text{Fixed value}} - \overbrace{\left[\frac{V_m I_m}{2}\cos(2\omega t + \theta_v + \theta_i)\right]}^{\text{Time-varying (function of } t)}$$

A plot of v, i, and p on the same set of axes is shown in Fig. 14.32.

Note that the second factor in the preceding equation is a cosine wave with an amplitude of $V_m I_m/2$ and with a frequency twice that of the voltage or current. The average value of this term is zero over one cycle, producing no net transfer of energy in any one direction.

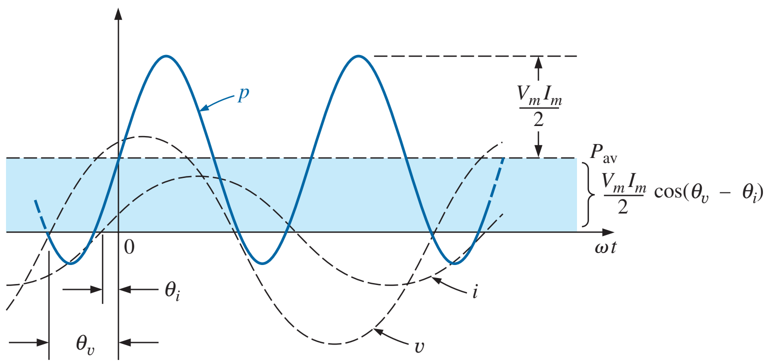

FIG. 14.32
Defining the average power for a sinusoidal ac network.

The first term in the preceding equation, however, has a constant magnitude (no time dependence) and therefore provides some net transfer of energy. This term is referred to as the **average power** or **real power** as introduced earlier. The angle $(\theta_v - \theta_i)$ is the phase angle between v and i. Since $\cos(-\alpha) = \cos\alpha$,

the magnitude of average power delivered is independent of whether v leads i or i leads v.

Defining θ as equal to $|\theta_v - \theta_i|$, where $|\ |$ indicates that only the magnitude is important and the sign is immaterial, we have

$$P = \frac{V_m I_m}{2}\cos\theta \qquad \text{(watts, W)} \qquad \textbf{(14.11)}$$

where P is the average power in watts. This equation can also be written

$$P = \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right)\cos\theta$$

or, since

$$V_{\text{eff}} = \frac{V_m}{\sqrt{2}} \quad \text{and} \quad I_{\text{eff}} = \frac{I_m}{\sqrt{2}}$$

Eq. (14.11) becomes

$$P = V_{\text{rms}} I_{\text{rms}} \cos\theta \qquad \textbf{(14.12)}$$

Let us now apply Eqs. (14.11) and (14.12) to the basic R, L, and C elements.

Resistor

In a purely resistive circuit, since υ and i are in phase, $|\theta_\upsilon - \theta_i| = \theta = 0°$, and $\cos\theta = \cos 0° = 1$, so that

$$P = \frac{V_m I_m}{2} = V_{\text{rms}} I_{\text{rms}} \quad \text{(W)} \tag{14.13}$$

Or, since

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{R}$$

then

$$P = \frac{V_{\text{rms}}^2}{R} = I_{\text{rms}}^2 R \quad \text{(W)} \tag{14.14}$$

Inductor

In a purely inductive circuit, since υ leads i by 90°, $|\theta_\upsilon - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2}\cos 90° = \frac{V_m I_m}{2}(0) = \mathbf{0\ W}$$

The average power or power dissipated by the ideal inductor (no associated resistance) is zero watts.

Capacitor

In a purely capacitive circuit, since i leads υ by 90°, $|\theta_\upsilon - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2}\cos(90°) = \frac{V_m I_m}{2}(0) = \mathbf{0\ W}$$

The average power or power dissipated by the ideal capacitor (no associated resistance) is zero watts.

EXAMPLE 14.10 Find the average power dissipated in a network whose input current and voltage are the following:

$$i = 5\sin(\omega t + 40°)$$
$$\upsilon = 10\sin(\omega t + 40°)$$

Solution: Since υ and i are in phase, the circuit appears to be purely resistive at the input terminals. Therefore,

$$P = \frac{V_m I_m}{2} = \frac{(10\ \text{V})(5\ \text{A})}{2} = \mathbf{25\ W}$$

or

$$R = \frac{V_m}{I_m} = \frac{10\ \text{V}}{5\ \text{A}} = 2\ \Omega$$

and

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{[(0.707)(10\ \text{V})]^2}{2} = \mathbf{25\ W}$$

or

$$P = I_{\text{rms}}^2 R = [(0.707)(5\ \text{A})]^2(2) = \mathbf{25\ W}$$

For the following example, the circuit consists of a combination of resistances and reactances producing phase angles between the input current and voltage different from 0° or 90°.

EXAMPLE 14.11 Determine the average power delivered to networks having the following input voltage and current:

a. $v = 100 \sin(\omega t + 40°)$
   $i = 20 \sin(\omega t + 70°)$
b. $v = 150 \sin(\omega t - 70°)$
   $i = 3 \sin(\omega t - 50°)$

Solutions:

a. $V_m = 100, \quad \theta_v = 40°$
   $I_m = 20 \text{ A}, \quad \theta_i = 70°$
   $\theta = |\theta_v - \theta_i| = |40° - 70°| = |-30°| = 30°$

and

$$P = \frac{V_m I_m}{2} \cos\theta = \frac{(100 \text{ V})(20 \text{ A})}{2} \cos(30°) = (1000 \text{ W})(0.866)$$
$$= \mathbf{866\ W}$$

b. $V_m = 150 \text{ V}, \quad \theta_v = -70°$
   $I_m = 3 \text{ A}, \quad \theta_i = -50°$
   $\theta = |\theta_v - \theta_i| = |-70° - (-50°)|$
   $= |-70° + 50°| = |-20°| = 20°$

and

$$P = \frac{V_m I_m}{2} \cos\theta = \frac{(150 \text{ V})(3 \text{ A})}{2} \cos(20°) = (225 \text{ W})(0.9397)$$
$$= \mathbf{211.43\ W}$$

Power Factor

In the equation $P = (V_m I_m/2)\cos\theta$, the factor that has significant control over the delivered power level is the $\cos\theta$. No matter how large the voltage or current, if $\cos\theta = 0$, the power is zero; if $\cos\theta = 1$, the power delivered is a maximum. Since it has such control, the expression was given the name **power factor** and is defined by

$$\text{Power factor} = F_p = \cos\theta \tag{14.15}$$

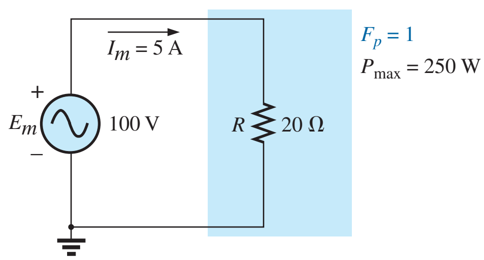

FIG. 14.33
Purely resistive load with $F_p = 1$.

For a purely resistive load such as the one shown in Fig. 14.33, the phase angle between v and i is 0° and $F_p = \cos\theta = \cos 0° = 1$. The power delivered is a maximum of $(V_m I_m/2)\cos\theta = ((100 \text{ V})(5 \text{ A})/2)(1) = 250$ W.

For a purely reactive load (inductive or capactitive) such as the one shown in Fig. 14.34, the phase angle between v and i is 90° and $F_p = \cos\theta = \cos 90° = 0$. The power delivered is then the minimum value of zero watts, *even though the current has the same peak value* as that encountered in Fig. 14.33.

For situations where the load is a combination of resistive and reactive elements, the power factor varies between 0 and 1. The more resistive the total impedance, the closer the power factor is to 1; the more reactive the total impedance, the closer the power factor is to 0.

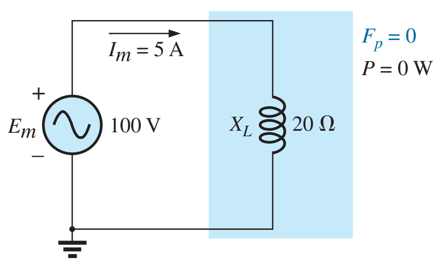

FIG. 14.34
Purely inductive load with $F_p = 0$.

In terms of the average power and the terminal voltage and current,

$$F_p = \cos\theta = \frac{P}{V_{\text{rms}} I_{\text{rms}}} \qquad \textbf{(14.16)}$$

The terms *leading* and *lagging* are often written in conjunction with the power factor. *They are defined by the current through the load.* If the current leads the voltage across a load, the load has a **leading power factor.** If the current lags the voltage across the load, the load has a **lagging power factor.** In other words,

capacitive networks have leading power factors, and inductive networks have lagging power factors.

The importance of the power factor to power distribution systems is examined in Chapter 19. In fact, one section is devoted to power-factor correction.

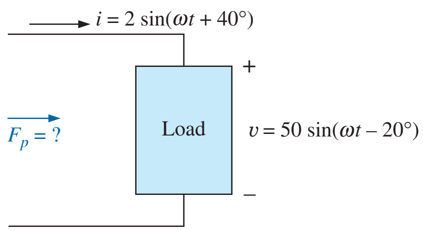

FIG. 14.35
Example 14.12(a).

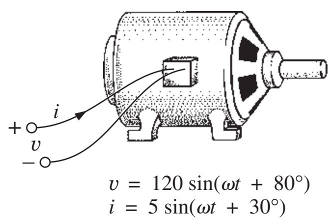

FIG. 14.36
Example 14.12(b).

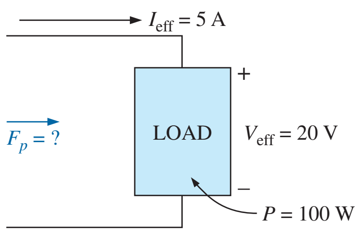

FIG. 14.37
Example 14.12(c).

EXAMPLE 14.12 Determine the power factors of the following loads, and indicate whether they are leading or lagging:

a. Fig. 14.35
b. Fig. 14.36
c. Fig. 14.37

Solutions:

a. $F_p = \cos\theta = \cos|40° - (-20°)| = \cos 60° = \mathbf{0.5\ leading}$
b. $F_p = \cos\theta\ |80° - 30°| = \cos 50° = \mathbf{0.64\ lagging}$
c. $F_p = \cos\theta = \dfrac{P}{V_{\text{eff}} I_{\text{eff}}} = \dfrac{100\text{ W}}{(20\text{ V})(5\text{ A})} = \dfrac{100\text{ W}}{100\text{ W}} = \mathbf{1}$

The load is resistive, and F_p is neither leading nor lagging.

14.6 COMPLEX NUMBERS

In our analysis of dc networks, we found it necessary to determine the algebraic sum of voltages and currents. Since the same will also be true for ac networks, the question arises, How do we determine the algebraic sum of two or more voltages (or currents) that are varying sinusoidally? Although one solution would be to find the algebraic sum on a point-to-point basis (as shown in Section 14.12), this would be a long and tedious process in which accuracy would be directly related to the scale used.

It is the purpose of this chapter to introduce a system of **complex numbers** that, when related to the sinusoidal ac waveform, results in a technique for finding the algebraic sum of sinusoidal waveforms that is quick, direct, and accurate. In the following chapters, the technique is extended to permit the analysis of sinusoidal ac networks in a manner very similar to that applied to dc networks. The methods and theorems as described for dc networks can then be applied to sinusoidal ac networks with little difficulty.

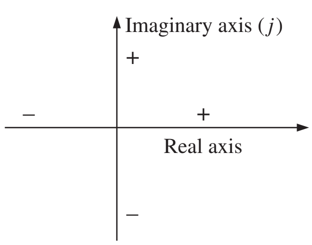

FIG. 14.38
Defining the real and imaginary axes of a complex plane.

A **complex number** represents a point in a two-dimensional plane located with reference to two distinct axes. This point can also determine a radius vector drawn from the origin to the point. The horizontal axis is called the *real* axis, while the vertical axis is called the *imaginary* axis. Both are labeled in Fig. 14.38. Every number from zero to $\pm\infty$ can be represented by some point along the real axis. Prior to the development of this system of complex numbers, it was believed that any number not on the real axis did not exist—hence the term *imaginary* for the vertical axis.

In the complex plane, the horizontal or real axis represents all positive numbers to the right of the imaginary axis and all negative numbers to the left of the imaginary axis. All positive imaginary numbers are represented above the real axis, and all negative imaginary numbers, below the real axis. The symbol j (or sometimes i) is used to denote the imaginary component.

Two forms are used to represent a complex number: **rectangular** and **polar.** Each can represent a point in the plane or a radius vector drawn from the origin to that point.

14.7 RECTANGULAR FORM

The format for the **rectangular form** is

$$\mathbf{C} = X + jY \tag{14.17}$$

as shown in Fig. 14.39. The letter **C** was chosen from the word "complex." The **boldface** notation is for any number with magnitude and direction. The *italic* is for magnitude only.

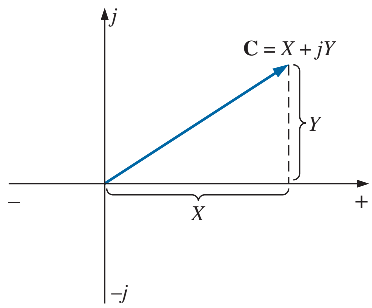

FIG. 14.39
Defining the rectangular form.

EXAMPLE 14.13 Sketch the following complex numbers in the complex plane:

a. $\mathbf{C} = 3 + j4$
b. $\mathbf{C} = 0 - j6$
c. $\mathbf{C} = -10 - j20$

Solutions:

a. See Fig. 14.40.
b. See Fig. 14.41.
c. See Fig. 14.42.

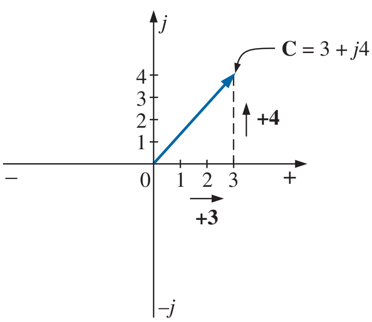

FIG. 14.40
Example 14.13(a).

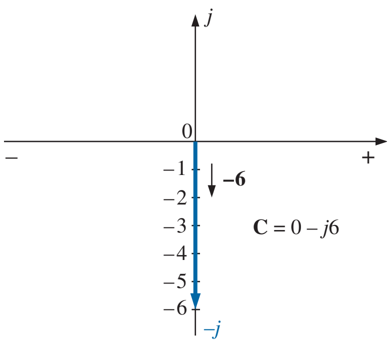

FIG. 14.41
Example 14.13(b).

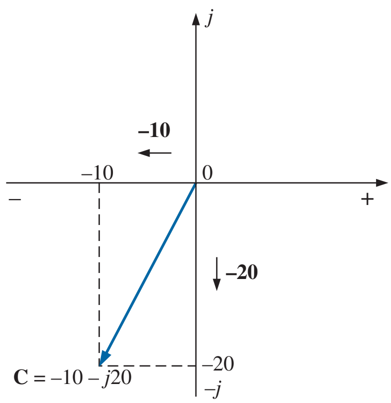

FIG. 14.42
Example 14.13(c).

14.8 POLAR FORM

The format for the **polar form** is

$$\mathbf{C} = Z \angle \theta \tag{14.18}$$

with the letter Z chosen from the sequence X, Y, Z.

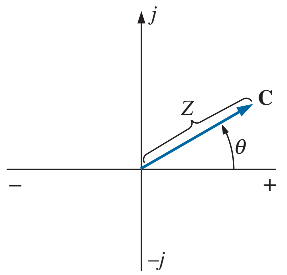

FIG. 14.43
Defining the polar form.

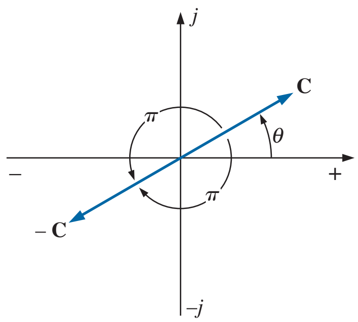

FIG. 14.44
Demonstrating the effect of a negative sign on the polar form.

Z indicates magnitude only and θ is *always measured counterclockwise (CCW) from the positive real axis,* as shown in Fig. 14.43. Angles measured in the clockwise direction from the positive real axis must have a negative sign associated with them.

A negative sign in front of the polar form has the effect shown in Fig. 14.44. Note that it results in a complex number directly opposite the complex number with a positive sign.

$$-\mathbf{C} = -Z\angle\theta = Z\angle\theta \pm 180^\circ \tag{14.19}$$

EXAMPLE 14.14 Sketch the following complex numbers in the complex plane:

a. $\mathbf{C} = 5\angle 30^\circ$
b. $\mathbf{C} = 7\angle -120^\circ$
c. $\mathbf{C} = -4.2\angle 60^\circ$

Solutions:

a. See Fig. 14.45.
b. See Fig. 14.46.
c. See Fig. 14.47.

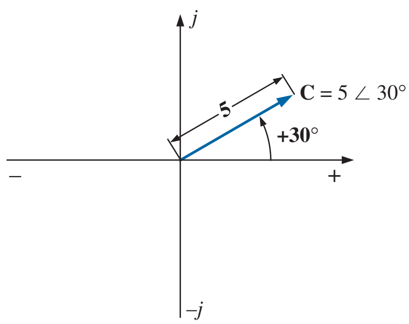

FIG. 14.45
Example 14.14(a).

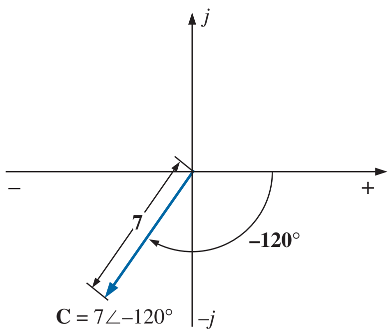

FIG. 14.46
Example 14.14(b).

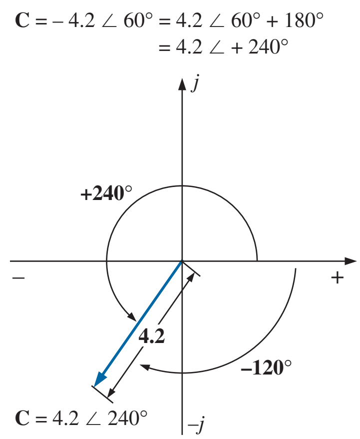

FIG. 14.47
Example 14.14(c).

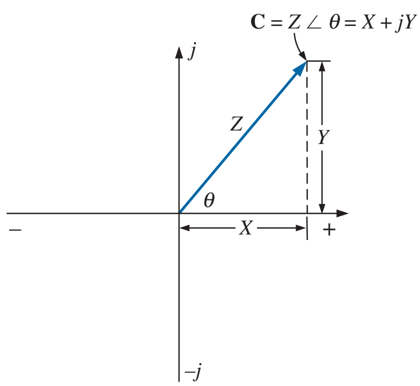

FIG. 14.48
Conversion between forms.

14.9 CONVERSION BETWEEN FORMS

The two forms are related by the following equations, as illustrated in Fig. 14.48.

Rectangular to Polar

$$Z = \sqrt{X^2 + Y^2} \tag{14.20}$$

$$\theta = \tan^{-1}\frac{Y}{X} \tag{14.21}$$

Polar to Rectangular

$$\boxed{X = Z\cos\theta} \tag{14.22}$$

$$\boxed{Y = Z\sin\theta} \tag{14.23}$$

EXAMPLE 14.15 Convert the following from rectangular to polar form:

$$\mathbf{C} = 3 + j4 \qquad \text{(Fig. 14.49)}$$

Solution:

$$Z = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ$$

and $$\mathbf{C = 5\angle 53.13^\circ}$$

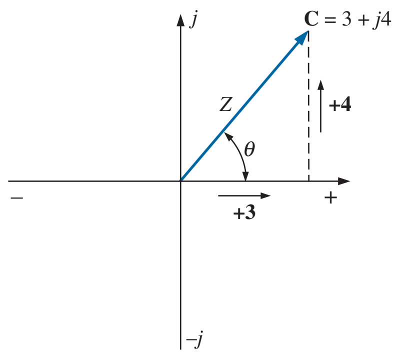

FIG. 14.49
Example 14.15.

EXAMPLE 14.16 Convert the following from polar to rectangular form:

$$\mathbf{C} = 10\angle 45^\circ \qquad \text{(Fig. 14.50)}$$

Solution:

$$X = 10\cos 45^\circ = (10)(0.707) = 7.07$$
$$Y = 10\sin 45^\circ = (10)(0.707) = 7.07$$

and $$\mathbf{C = 7.07 + \mathit{j}7.07}$$

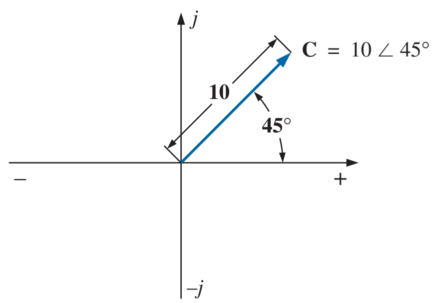

FIG. 14.50
Example 14.16.

If the complex number should appear in the second, third, or fourth quadrant, simply convert it in that quadrant, and carefully determine the proper angle to be associated with the magnitude of the vector.

EXAMPLE 14.17 Convert the following from rectangular to polar form:

$$\mathbf{C} = -6 + j3 \qquad \text{(Fig. 14.51)}$$

Solution:

$$Z = \sqrt{(6)^2 + (3)^2} = \sqrt{45} = 6.71$$

$$\beta = \tan^{-1}\left(\frac{3}{6}\right) = 26.57^\circ$$

$$\theta = 180^\circ - 26.57^\circ = 153.43^\circ$$

and $$\mathbf{C = 6.71\angle 153.43^\circ}$$

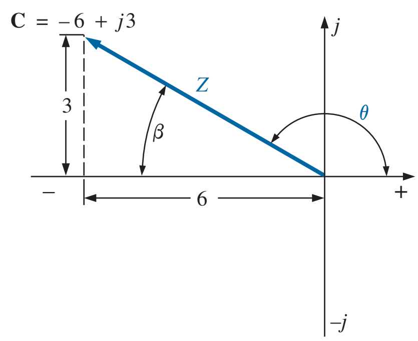

FIG. 14.51
Example 14.17.

EXAMPLE 14.18 Convert the following from polar to rectangular form:

$$\mathbf{C} = 10\angle 230^\circ \qquad \text{(Fig. 14.52)}$$

Solution:

$$X = Z\cos\beta = 10\cos(230^\circ - 180^\circ) = 10\cos 50^\circ$$
$$= (10)(0.6428) = 6.428$$
$$Y = Z\sin\beta = 10\sin 50^\circ = (10)(0.7660) = 7.660$$

and $$\mathbf{C = -6.43 - \mathit{j}7.66}$$

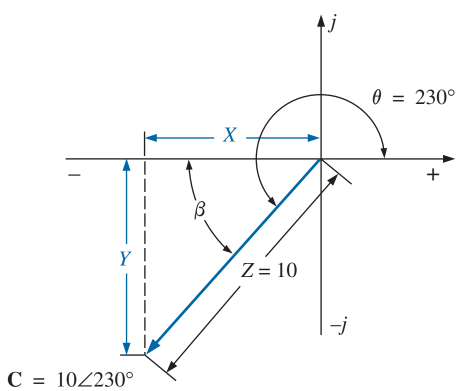

FIG. 14.52
Example 14.18.

14.10 MATHEMATICAL OPERATIONS WITH COMPLEX NUMBERS

Complex numbers lend themselves readily to the basic mathematical operations of addition, subtraction, multiplication, and division. A few basic rules and definitions must be understood before considering these operations.

Let us first examine the symbol j associated with imaginary numbers. By definition,

$$j = \sqrt{-1} \tag{14.24}$$

Thus,

$$j^2 = -1 \tag{14.25}$$

and $$j^3 = j^2 j = -1j = -j$$

with $$j^4 = j^2 j^2 = (-1)(-1) = +1$$
$$j^5 = j$$

and so on. Further,

$$\frac{1}{j} = (1)\left(\frac{1}{j}\right) = \left(\frac{j}{j}\right)\left(\frac{1}{j}\right) = \frac{j}{j^2} = \frac{j}{-1}$$

and

$$\frac{1}{j} = -j \tag{14.26}$$

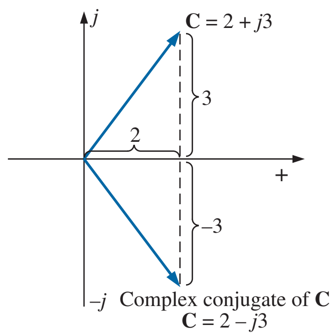

FIG. 14.53
Defining the complex conjugate of a complex number in rectangular form.

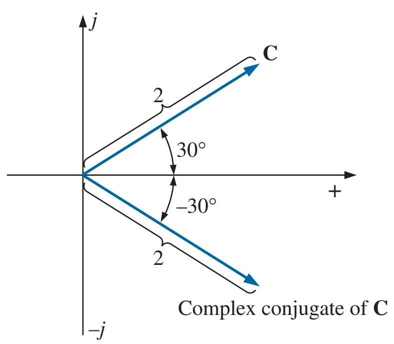

FIG. 14.54
Defining the complex conjugate of a complex number in polar form.

Complex Conjugate

The **conjugate** or **complex conjugate** of a complex number can be found by simply changing the sign of the imaginary part in the rectangular form or by using the negative of the angle of the polar form. For example, the conjugate of

$$\mathbf{C} = 2 + j3$$

is $$2 - j3$$

as shown in Fig. 14.53. The conjugate of

$$\mathbf{C} = 2 \angle 30^\circ$$

is $$2 \angle -30^\circ$$

as shown in Fig. 14.54.

Reciprocal

The **reciprocal** of a complex number is 1 divided by the complex number. For example, the reciprocal of

$$\mathbf{C} = X + jY$$

is $$\frac{1}{X + jY}$$

and of $Z \angle \theta$,

$$\frac{1}{Z \angle \theta}$$

We are now prepared to consider the four basic operations of *addition, subtraction, multiplication,* and *division* with complex numbers.

Addition

To add two or more complex numbers, add the real and imaginary parts separately. For example, if

$$\mathbf{C}_1 = \pm X_1 \pm jY_1 \quad \text{and} \quad \mathbf{C}_2 = \pm X_2 \pm jY_2$$

then

$$\mathbf{C}_1 + \mathbf{C}_2 = (\pm X_1 \pm X_2) + j(\pm Y_1 \pm Y_2) \tag{14.27}$$

There is really no need to memorize the equation. Simply set one above the other and consider the real and imaginary parts separately, as shown in Example 14.19.

EXAMPLE 14.19

a. Add $\mathbf{C}_1 = 2 + j4$ and $\mathbf{C}_2 = 3 + j1$.
b. Add $\mathbf{C}_1 = 3 + j6$ and $\mathbf{C}_2 = -6 + j3$.

Solutions:

a. By Eq. (14.27),

$$\mathbf{C}_1 + \mathbf{C}_2 = (2 + 3) + j(4 + 1) = \mathbf{5 + j5}$$

Note Fig. 14.55. An alternative method is

$$\begin{array}{c} 2 + j4 \\ \underline{3 + j1} \\ \downarrow \quad \downarrow \\ \mathbf{5 + j5} \end{array}$$

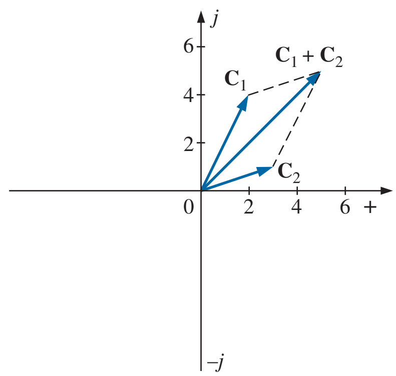

FIG. 14.55
Example 14.19(a).

b. By Eq. (14.27),

$$\mathbf{C}_1 + \mathbf{C}_2 = (3 - 6) + j(6 + 3) = \mathbf{-3 + j9}$$

Note Fig. 14.56. An alternative method is

$$\begin{array}{c} 3 + j6 \\ \underline{-6 + j3} \\ \downarrow \quad \downarrow \\ \mathbf{-3 + j9} \end{array}$$

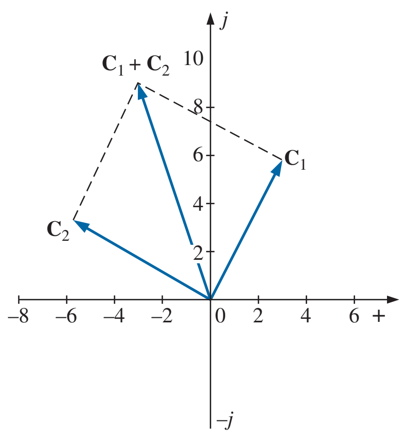

FIG. 14.56
Example 14.19(b).

Subtraction

In subtraction, the real and imaginary parts are again considered separately. For example, if

$$\mathbf{C}_1 = \pm X_1 \pm jY_1 \quad \text{and} \quad \mathbf{C}_2 = \pm X_2 \pm jY_2$$

then

$$\mathbf{C}_1 - \mathbf{C}_2 = [\pm X_1 - (\pm X_2)] + j[\pm Y_1 - (\pm Y_2)] \tag{14.28}$$

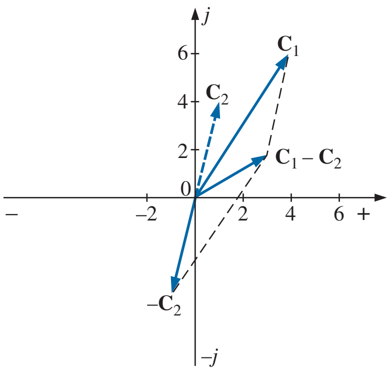

FIG. 14.57
Example 14.20(a).

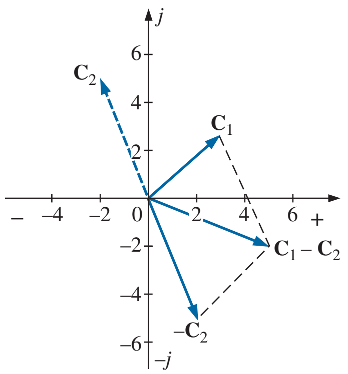

FIG. 14.58
Example 14.20(b).

Again, there is no need to memorize the equation if the alternative method of Example 14.20 is used.

EXAMPLE 14.20

a. Subtract $\mathbf{C}_2 = 1 + j4$ from $\mathbf{C}_1 = 4 + j6$.
b. Subtract $\mathbf{C}_2 = -2 + j5$ from $\mathbf{C}_1 = +3 + j3$.

Solutions:

a. By Eq. (14.28),

$$\mathbf{C}_1 - \mathbf{C}_2 = (4 - 1) + j(6 - 4) = \mathbf{3 + \mathit{j}2}$$

Note Fig. 14.57. An alternative method is

$$\begin{array}{c} 4 + j6 \\ \underline{-(1 + j4)} \\ \downarrow \quad \downarrow \\ \mathbf{3 + \mathit{j}2} \end{array}$$

b. By Eq. (14.28),

$$\mathbf{C}_1 - \mathbf{C}_2 = [3 - (-2)] + j(3 - 5) = \mathbf{5 - \mathit{j}2}$$

Note Fig. 14.58. An alternative method is

$$\begin{array}{c} 3 + j3 \\ \underline{-(-2 + j5)} \\ \downarrow \quad \downarrow \\ \mathbf{5 - \mathit{j}2} \end{array}$$

Addition or subtraction cannot be performed in polar form unless the complex numbers have the same angle θ or unless they differ only by multiples of 180°.

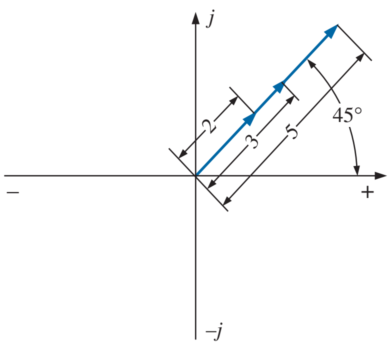

FIG. 14.59
Example 14.21(a).

EXAMPLE 14.21

a. $2 \angle 45° + 3 \angle 45° = \mathbf{5 \angle 45°}$. Note Fig. 14.59.

b. $2 \angle 0° - 4 \angle 180° = \mathbf{6 \angle 0°}$. Note Fig. 14.60.

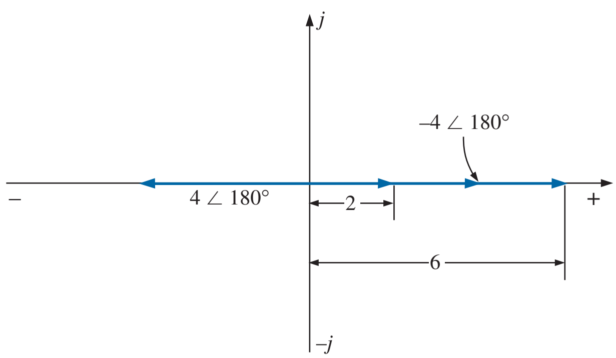

FIG. 14.60
Example 14.21(b).

Multiplication

To multiply two complex numbers in *rectangular* form, multiply the real and imaginary parts of one in turn by the real and imaginary parts of the other. For example, if

$$\mathbf{C}_1 = X_1 + jY_1 \quad \text{and} \quad \mathbf{C}_2 = X_2 + jY_2$$

then $\mathbf{C}_1 \cdot \mathbf{C}_2$:

$$\begin{array}{r} X_1 + jY_1 \\ \underline{X_2 + jY_2} \\ X_1X_2 + jY_1X_2 \\ \underline{\qquad + jX_1Y_2 + j^2Y_1Y_2} \\ X_1X_2 + j(Y_1X_2 + X_1Y_2) + Y_1Y_2(-1) \end{array}$$

and

$$\boxed{\mathbf{C}_1 \cdot \mathbf{C}_2 = (X_1X_2 - Y_1Y_2) + j(Y_1X_2 + X_1Y_2)} \tag{14.29}$$

In Example 14.22(b), we obtain a solution without resorting to memorizing Eq. (14.29). Simply carry along the j factor when multiplying each part of one vector with the real and imaginary parts of the other.

EXAMPLE 14.22

a. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = 2 + j3 \quad \text{and} \quad \mathbf{C}_2 = 5 + j10$$

b. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = -2 - j3 \quad \text{and} \quad \mathbf{C}_2 = +4 - j6$$

Solutions:

a. Using the format above, we have

$$\begin{aligned} \mathbf{C}_1 \cdot \mathbf{C}_2 &= [(2)(5) - (3)(10)] + j[(3)(5) + (2)(10)] \\ &= \mathbf{-20 + j35} \end{aligned}$$

b. Without using the format, we obtain

$$\begin{array}{l} -2 - j3 \\ \underline{+4 - j6} \\ -8 - j12 \\ \underline{\quad + j12 + j^2 18} \\ -8 + j(-12 + 12) - 18 \end{array}$$

and $\mathbf{C}_1 \cdot \mathbf{C}_2 = \mathbf{-26 = 26 \angle 180°}$

In *polar* form, the magnitudes are multiplied and the angles added algebraically. For example, for

$$\mathbf{C}_1 = Z_1 \angle \theta_1 \quad \text{and} \quad \mathbf{C}_2 = Z_2 \angle \theta_2$$

we write

$$\boxed{\mathbf{C}_1 \cdot \mathbf{C}_2 = Z_1Z_2 \angle \theta_1 + \theta_2} \tag{14.30}$$

EXAMPLE 14.23

a. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = 5 \angle 20° \quad \text{and} \quad \mathbf{C}_2 = 10 \angle 30°$$

b. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = 2 \angle -40^\circ \quad \text{and} \quad \mathbf{C}_2 = 7 \angle +120^\circ$$

Solutions:

a. $\mathbf{C}_1 \cdot \mathbf{C}_2 = (5 \angle 20^\circ)(10 \angle 30^\circ) = (5)(10) \underline{/20^\circ + 30^\circ} = \mathbf{50 \angle 50^\circ}$

b. $\mathbf{C}_1 \cdot \mathbf{C}_2 = (2 \angle -40^\circ)(7 \angle +120^\circ) = (2)(7) \underline{/-40^\circ + 120^\circ}$
$= \mathbf{14 \angle +80^\circ}$

To multiply a complex number in rectangular form by a real number requires that both the real part and the imaginary part be multiplied by the real number. For example,

$$(10)(2 + j3) = 20 + j30$$

and

$$50 \angle 0^\circ \, (0 + j6) = j300 = 300 \angle 90^\circ$$

Division

To divide two complex numbers in *rectangular* form, multiply the numerator and denominator by the conjugate of the denominator and the resulting real and imaginary parts collected. That is, if

$$\mathbf{C}_1 = X_1 + jY_1 \quad \text{and} \quad \mathbf{C}_2 = X_2 + jY_2$$

then

$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{(X_1 + jY_1)(X_2 - jY_2)}{(X_2 + jY_2)(X_2 - jY_2)}$$

$$= \frac{(X_1X_2 + Y_1Y_2) + j(X_2Y_1 - X_1Y_2)}{X_2^2 + Y_2^2}$$

and

$$\boxed{\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{X_1X_2 + Y_1Y_2}{X_2^2 + Y_2^2} + j\frac{X_2Y_1 - X_1Y_2}{X_2^2 + Y_2^2}} \qquad \mathbf{(14.31)}$$

The equation does not have to be memorized if the steps above used to obtain it are employed. That is, first multiply the numerator by the complex conjugate of the denominator and separate the real and imaginary terms. Then divide each term by the sum of each term of the denominator squared.

EXAMPLE 14.24

a. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 1 + j4$ and $\mathbf{C}_2 = 4 + j5$.
b. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = -4 - j8$ and $\mathbf{C}_2 = +6 - j1$.

Solutions:

a. By Eq. (14.31),

$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{(1)(4) + (4)(5)}{4^2 + 5^2} + j\frac{(4)(4) - (1)(5)}{4^2 + 5^2}$$

$$= \frac{24}{41} + \frac{j11}{41} \cong \mathbf{0.59 + j0.27}$$

b. Using an alternative method, we obtain

$$\begin{array}{l} -4 - j8 \\ \underline{+6 + j1} \\ -24 - j48 \\ \underline{\quad - j4 - j^2 8} \\ -24 - j52 + 8 = -16 - j52 \end{array}$$

$$\begin{array}{l} +6 - j1 \\ \underline{+6 + j1} \\ 36 + j6 \\ \underline{\quad - j6 - j^2 1} \\ 36 + \ \ 0 + 1 \ = 37 \end{array}$$

and $$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{-16}{37} - \frac{j52}{37} = \mathbf{-0.43 - j1.41}$$

To divide a complex number in rectangular form by a real number, both the real part and the imaginary part must be divided by the real number. For example,

$$\frac{8 + j10}{2} = 4 + j5$$

and $$\frac{6.8 - j0}{2} = 3.4 - j0 = 3.4\angle 0^\circ$$

In *polar* form, division is accomplished by dividing the magnitude of the numerator by the magnitude of the denominator and subtracting the angle of the denominator from that of the numerator. That is, for

$$\mathbf{C}_1 = Z_1 \angle\theta_1 \quad \text{and} \quad \mathbf{C}_2 = Z_2 \angle\theta_2$$

we write

$$\boxed{\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{Z_1}{Z_2}\angle\theta_1 - \theta_2} \qquad \textbf{(14.32)}$$

EXAMPLE 14.25

a. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 15\angle 10^\circ$ and $\mathbf{C}_2 = 2\angle 7^\circ$.
b. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 8\angle 120^\circ$ and $\mathbf{C}_2 = 16\angle -50^\circ$.

Solutions:

a. $$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{15\angle 10^\circ}{2\angle 7^\circ} = \frac{15}{2}\angle 10^\circ - 7^\circ = \mathbf{7.5\angle 3^\circ}$$

b. $$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{8\angle 120^\circ}{16\angle -50^\circ} = \frac{8}{16}\angle 120^\circ - (-50^\circ) = \mathbf{0.5\angle 170^\circ}$$

We obtain the *reciprocal* in the rectangular form by multiplying the numerator and denominator by the complex conjugate of the denominator:

$$\frac{1}{X + jY} = \left(\frac{1}{X + jY}\right)\left(\frac{X - jY}{X - jY}\right) = \frac{X - jY}{X^2 + Y^2}$$

and

$$\frac{1}{X + jY} = \frac{X}{X^2 + Y^2} - j\frac{Y}{X^2 + Y^2} \qquad (14.33)$$

In polar form, the reciprocal is

$$\frac{1}{Z\angle\theta} = \frac{1}{Z}\angle -\theta \qquad (14.34)$$

A concluding example using the four basic operations follows.

EXAMPLE 14.26 Perform the following operations, leaving the answer in polar or rectangular form:

a.
$$\begin{aligned}
\frac{(2 + j3) + (4 + j6)}{(7 + j7) - (3 - j3)} &= \frac{(2 + 4) + j(3 + 6)}{(7 - 3) + j(7 + 3)} \\
&= \frac{(6 + j9)(4 - j10)}{(4 + j10)(4 - j10)} \\
&= \frac{[(6)(4) + (9)(10)] + j[(4)(9) - (6)(10)]}{4^2 + 10^2} \\
&= \frac{114 - j24}{116} = \mathbf{0.98 - j0.21}
\end{aligned}$$

b.
$$\begin{aligned}
\frac{(50\angle 30^\circ)(5 + j5)}{10\angle -20^\circ} &= \frac{(50\angle 30^\circ)(7.07\angle 45^\circ)}{10\angle -20^\circ} = \frac{353.5\angle 75^\circ}{10\angle -20^\circ} \\
&= 35.35\,\angle 75^\circ - (-20^\circ) = \mathbf{35.35\angle 95^\circ}
\end{aligned}$$

c.
$$\begin{aligned}
\frac{(2\angle 20^\circ)^2(3 + j4)}{8 - j6} &= \frac{(2\angle 20^\circ)(2\angle 20^\circ)(5\angle 53.13^\circ)}{10\angle -36.87^\circ} \\
&= \frac{(4\angle 40^\circ)(5\angle 53.13^\circ)}{10\angle -36.87^\circ} = \frac{20\angle 93.13^\circ}{10\angle -36.87^\circ} \\
&= 2\,\angle 93.13^\circ - (-36.87^\circ) = \mathbf{2.0\angle 130^\circ}
\end{aligned}$$

d.
$$\begin{aligned}
3\angle 27^\circ - 6\angle -40^\circ &= (2.673 + j1.362) - (4.596 - j3.857) \\
&= (2.673 - 4.596) + j(1.362 + 3.857) \\
&= \mathbf{-1.92 + j5.22}
\end{aligned}$$

14.11 CALCULATOR AND COMPUTER METHODS WITH COMPLEX NUMBERS

The process of converting from one form to another or working through lengthy operations with complex numbers can be time-consuming and often frustrating if one lost minus sign or decimal point invalidates the solution. Fortunately, technologists of today have calculators and computer methods that make the process measurably easier with higher degrees of reliability and accuracy.

Calculators

The TI-89 calculator in Fig. 14.61 is only one of numerous calculators that can convert from one form to another and perform lengthy calcula-

FIG. 14.61
TI-89 scientific calculator.
(Courtesy of Texas Instruments, Inc.)

tions with complex numbers in a concise, neat form. Not all of the details of using a specific calculator are included here because each has its own format and sequence of steps. However, the basic operations with the TI-89 is included primarily to demonstrate the ease with which the conversions can be made and the format for more complex operations. If you have a TI-86 calculator, Appendix B provides details for using that calculator to perform these operations.

There are different routes to perform the conversions and operations below, but these instructions give you one approach that is fairly direct and straightforward. Since most operations are in the DEGREE rather than RADIAN mode, the sequence in Fig. 14.62 shows how to set the DEGREE mode for the operations to follow. A similar sequence sets the RADIAN mode if required.

MODE ↓ Angle → ↓ DEGREE ENTER ENTER

FIG. 14.62
Setting the DEGREE mode on the TI-89 calculator.

Rectangular to Polar Conversion The sequence in Fig. 14.63 provides a detailed listing of the steps needed to convert from rectangular to polar form. In the examples to follow, the scrolling steps are not listed to simplify the sequence.

In the sequence in Fig. 14.63, an up scroll is chosen after Matrix because that is a more direct path to Vector ops. A down scroll generates the same result, but it requires going through the whole listing. The sequence seems quite long for such a simple conversion, but with practice you will be able to perform the scrolling steps quite rapidly. Always be sure the input data is entered correctly, such as including the *i* after the *y* component. Any incorrect entry will result in an error listing.

(3 + 5 2ND *i*) 2ND MATH ↓ Matrix →
↑ Vector ops → ↓ ▶ Polar ENTER ENTER **5.83E0 ∠ 59.0E0**

FIG. 14.63
Converting 3 + j5 to the polar form using the TI-89 calculator.

Polar to Rectangular Conversion The sequence in Fig. 14.64 is a detailed listing of the steps needed to convert from polar to rectangular form. Note in the format that the brackets must surround the polar form. Also, the degree sign must be included with the angle to perform the calculation. The answer is displayed in the engineering notation selected.

(5 2ND ∠ 5 3 • 1 2ND MATH ↓
Angle → ° ENTER) 2ND MATH ↓ Matrix → ↑ Vector ops → ↓
Rect ENTER ENTER **3.00E0+4.00E0i**

FIG. 14.64
Converting 5∠53.1° to the rectangular form using the TI-89 calculator.

Mathematical Operations Mathematical operations are performed in the natural order of operations, but you must remember to select the format for the solution. For instance, if the sequence in Fig. 14.65 did not include the polar designation, the answer would be in rectangular form

(1 0 ∠ 5 0 °) × (2 ∠ 2 0 °)
▶ Polar ENTER **20.00E0 ∠ 70.00E0**

FIG. 14.65
Performing the operation (10∠50°)(2∠20°).

even though both quantities in the calculation are in polar form. In the rest of the examples, the scrolling required to obtain mathematical functions is not included to minimize the length of the sequence.

For the product of mixed complex numbers, the sequence of Fig. 14.66 results. Again, the polar form was selected for the solution.

(5 ∠ 5 3 . 1 °) × (2 + 2 *i*)
▶ Polar ENTER ENTER **14.14E0 ∠ 98.10E0**

FIG. 14.66
Performing the operation (5∠53.1°)(2 + j2).

Finally, Example 14.26(c) is entered as shown by the sequence in Fig. 14.67. Note that the results exactly match those obtained earlier.

(2 ∠ 2 0 °) ^ 2 × (3 + 4 *i*)
÷ (8 − 6 *i*) ▶ Polar ENTER ENTER **2.00E0 ∠ 130.0E0**

FIG. 14.67
Verifying the results of Example 14.26(c).

Mathcad

The Mathcad format for complex numbers is now introduced in preparation for the chapters to follow. We continue to use j when we define a complex number in rectangular form even though the Mathcad result always appears with the letter i. You can change this by going to the **Format** menu, but for this presentation we decided to use the default operators as much as possible.

When entering j to define the imaginary component of a complex number, be sure to enter it as 1j, but do not put a multiplication operator between the 1 and the j. Just type 1 and then j. In addition, place the j after the constant rather than before as in the text material.

When Mathcad operates on an angle, it assumes that the angle is in radians and not degrees. Further, all results appear in radians rather than degrees.

The first operation to be developed is the conversion from rectangular to polar form. In Fig. 14.68, the rectangular number $4 + j3$ is being converted to polar form using Mathcad. First define X and Y using the colon operator. Next, write the equation for the magnitude of the polar form in terms of the two variables just defined. The magnitude of the polar form is then revealed by writing the variable again and using the equal sign. It takes some practice, but be careful when writing the equation for Z; you must pay particular attention to the location of the bracket before performing the next operation. The resulting magnitude of 5 is as expected.

For the angle, the sequence **View-Toolbars-Greek** is first applied to obtain the **Greek** toolbar appearing in Fig. 14.68. It can be moved to any location by clicking on the blue at the top of the toolbar and dragging it

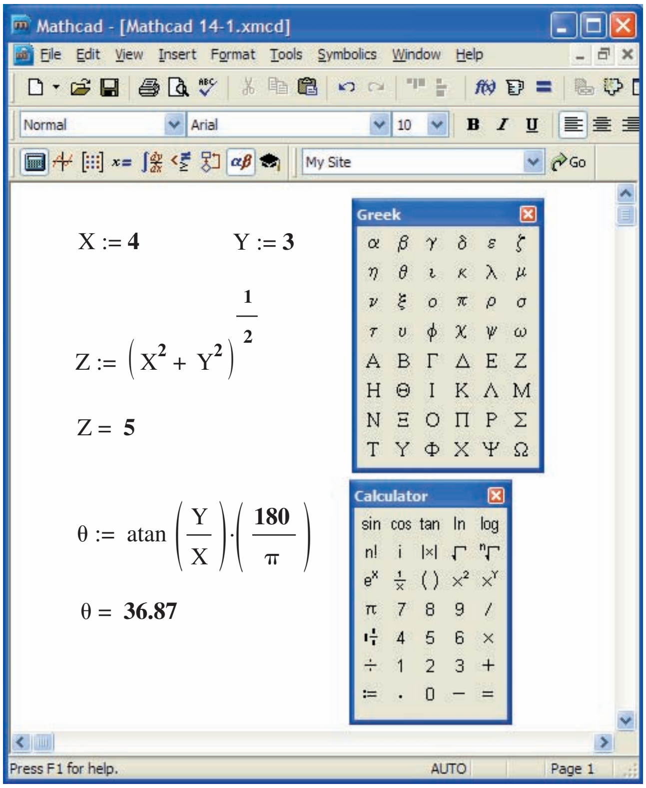

FIG. 14.68
Using Mathcad to convert from rectangular to polar form.

to the preferred location. Then select θ from the toolbar as the variable to be defined. Obtain the $\tan^{-1}\theta$ through the sequence **Insert-*f*(*x*)-Insert Function** dialog box**-trigonometric-atan-OK** in which *Y/X* is inserted. Then bring the controlling bracket to the outside of the entire expression, and multiply by the ratio of $180/\pi$ with π selected from the **Calculator** toolbar (available from the same sequence used to obtain the **Greek** toolbar). The multiplication by the last factor of the equation ensures that the angle is in degrees. Selecting θ again followed by an equal sign results in the correct angle of 36.87° as shown in Fig. 14.68.

We now look at two forms for the polar form of a complex number. The first is defined by the basic equations introduced in this chapter, while the second uses a special format. For all the Mathcad analyses to be provided in this text, the latter format is used. First define the magnitude of the polar form followed by the conversion of the angle of 60° to radians by multiplying by the factor $\pi/180$ as shown in Fig. 14.69. In this example, the resulting angular measure is $\pi/3$ radians. Next define the rectangular format by a real part $X = Z\cos\theta$ and by an imaginary part $Y = Z\sin\theta$. Both the cos and the sin are obtained by the sequence **Insert-*f*(*x*)-trigonometric-cos**(or **sin**)**-OK.** Note the multiplication by *j* which was actually entered as 1*j* without the multiplication operator between the 1 and the *j*. Entering *C* again followed by an equal sign results in the correct conversion shown in Fig. 14.69.

The next format is based on the mathematical relationship that $e^{j\theta} = \cos\theta + j\sin\theta$. Both *Z* and θ are as defined above, but now the complex number is written as shown in Fig. 14.69 using the notation just introduced. Note that both *Z* and θ are part of this defining form. The e^x is obtained directly from the **Calculator** toolbar. Remember to enter the *j* as

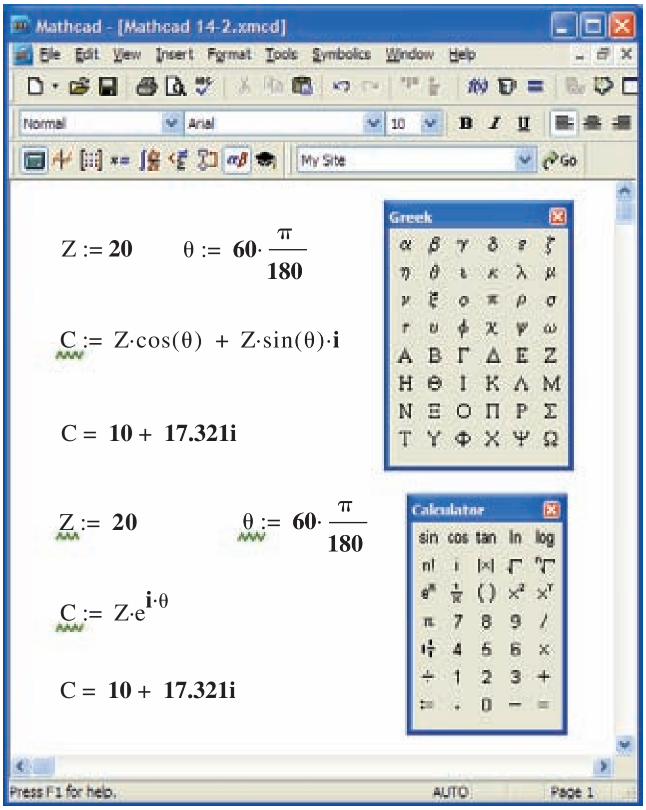

FIG. 14.69
Using Mathcad to convert from polar to rectangular form.

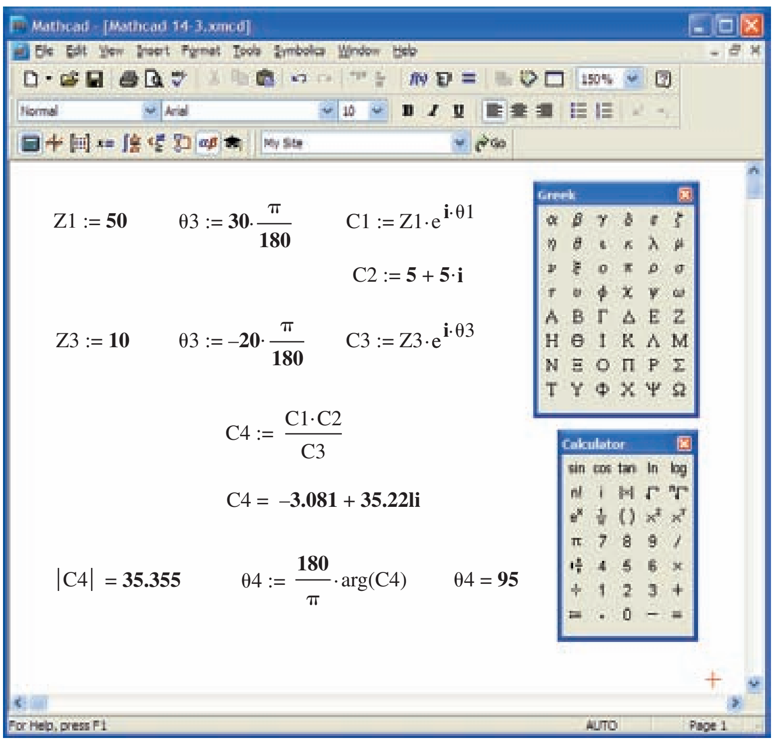

FIG. 14.70
Using Mathcad to confirm the results of Example 14.26(b).

$1j$ without a multiplication sign between the 1 and the j. However, there is a multiplication operator placed between the j and θ. When entered again followed by an equal sign, the rectangular form appears to match the above results. As mentioned above, it is this latter format that will be used throughout the text due to its cleaner form and more direct entering path.

The last example using Mathcad is a confirmation of the results of Example 14.26(b) as shown in Fig. 14.70. First define the three complex numbers as shown. Then enter the equation for the desired result using C_4, and the results are displayed. Note the relative simplicity of the equation for C_4 now that all the other variables have been defined. As shown, however, the immediate result is in the rectangular form. The components of the polar form can be obtained using the |×| function from the **Calculator** toolbar and the **arg** function from **Insert-*f(x)*-Complex Numbers-arg.** There are many other examples in the chapters to follow on the use of Mathcad with complex numbers.

14.12 PHASORS

As noted earlier in this chapter, the addition of sinusoidal voltages and currents is frequently required in the analysis of ac circuits. One lengthy but valid method of performing this operation is to place both sinusoidal waveforms on the same set of axes and add algebraically the magnitudes of each at every point along the abscissa, as shown for $c = a + b$ in Fig. 14.71. This, however, can be a long and tedious process with limited accuracy. A shorter method uses the rotating radius vector first appearing in Fig. 13.16. This *radius vector,* having a *constant magnitude* (length) with *one end fixed at the origin,* is called a **phasor** when applied to electric circuits. During its rotational development of the sine wave, the phasor will, at the instant $t = 0$, have the positions shown in Fig. 14.72(a) for each waveform in Fig. 14.72(b).

Note in Fig. 14.72(b) that v_2 passes through the horizontal axis at $t = 0$ s, requiring that the radius vector in Fig. 14.72(a) be on the horizontal axis to ensure a vertical projection of zero volts at $t = 0$ s. Its length in Fig. 14.72(a) is equal to the peak value of the sinusoid as required by the radius vector in Fig. 13.16. The other sinusoid has passed through 90° of its rotation by the time $t = 0$ s is reached and therefore has its maximum

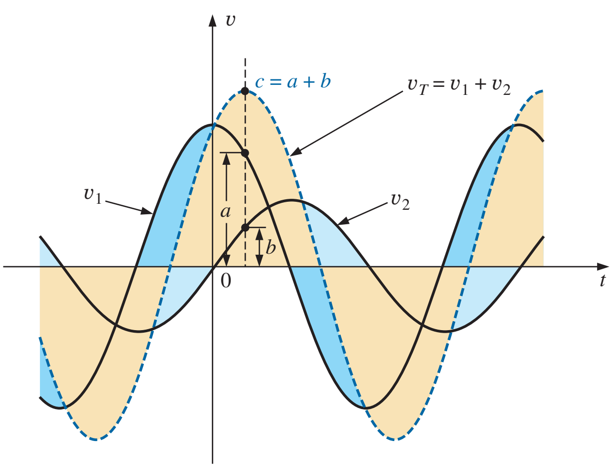

FIG. 14.71
Adding two sinusoidal waveforms on a point-by-point basis.

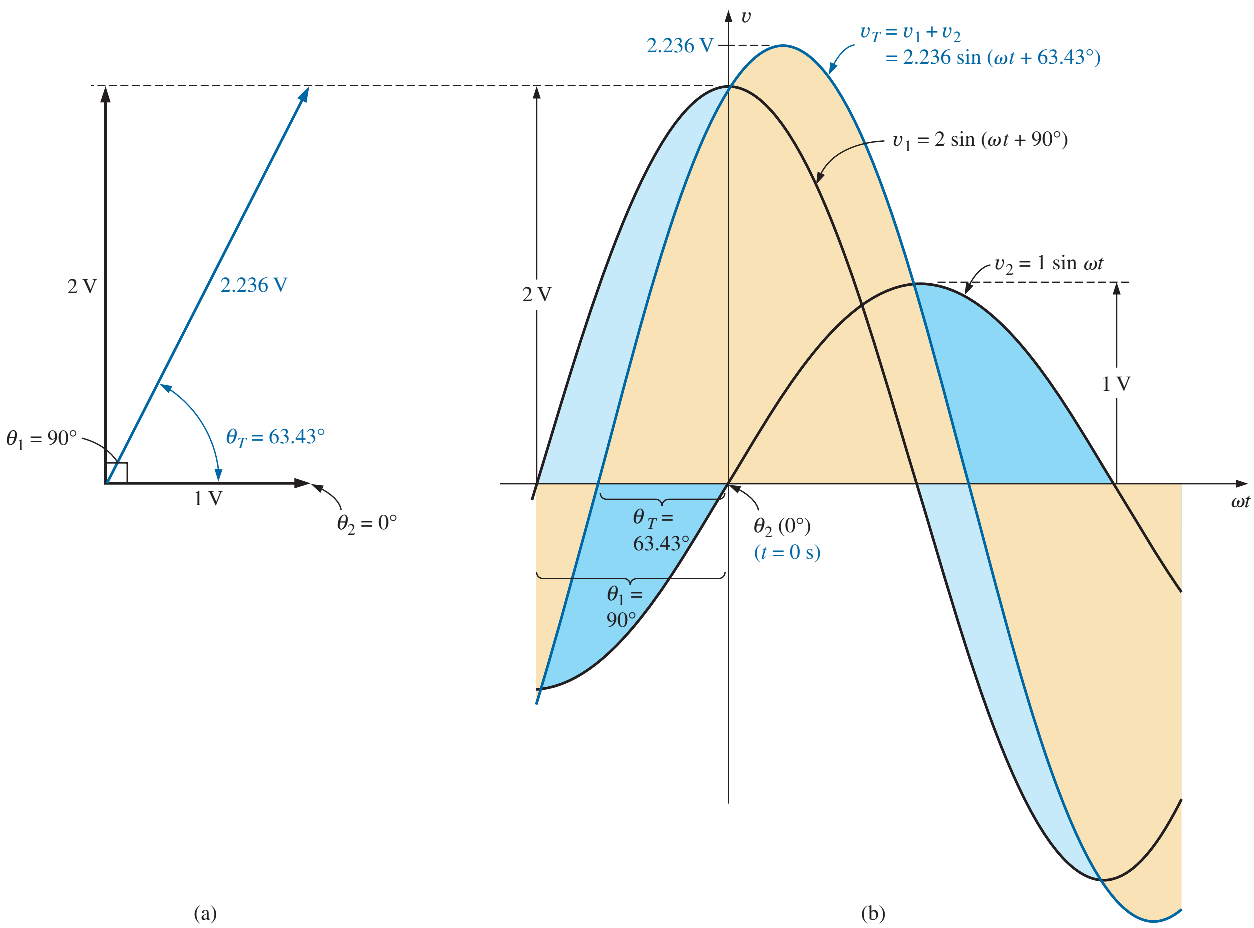

FIG. 14.72

(a) The phasor representation of the sinusoidal waveforms of Fig. 14.72(b); (b) finding the sum of two sinusoidal waveforms of v_1 and v_2.

vertical projection as shown in Fig. 14.72(a). Since the vertical projection is a maximum, the peak value of the sinusoid that it generates is also attained at $t = 0$ s, as shown in Fig. 14.72(b). Note also that $v_T = v_1$ at $t = 0$ s since $v_2 = 0$ V at this instant.

It can be shown [see Fig. 14.72(a)] using the vector algebra described in Section 14.10 that

$$1 \text{ V } \angle 0° + 2 \text{ V } \angle 90° = 2.236 \text{ V } \angle 63.43°$$

In other words, if we convert v_1 and v_2 to the phasor form using

$$v = V_m \sin(\omega t \pm \theta) \Rightarrow V_m \angle \pm\theta$$

and add them using complex number algebra, we can find the phasor form for v_T with very little difficulty. It can then be converted to the time domain and plotted on the same set of axes, as shown in Fig. 14.72(b). Fig. 14.72(a), showing the magnitudes and relative positions of the various phasors, is called a **phasor diagram.** It is actually a "snapshot" of the rotating radius vectors at $t = 0$ s.

In the future, therefore, if the addition of two sinusoids is required, you should first convert them to the phasor domain and find the sum using complex algebra. You can then convert the result to the time domain.

The case of two sinusoidal functions having phase angles different from 0° and 90° appears in Fig. 14.73. Note again that the vertical height

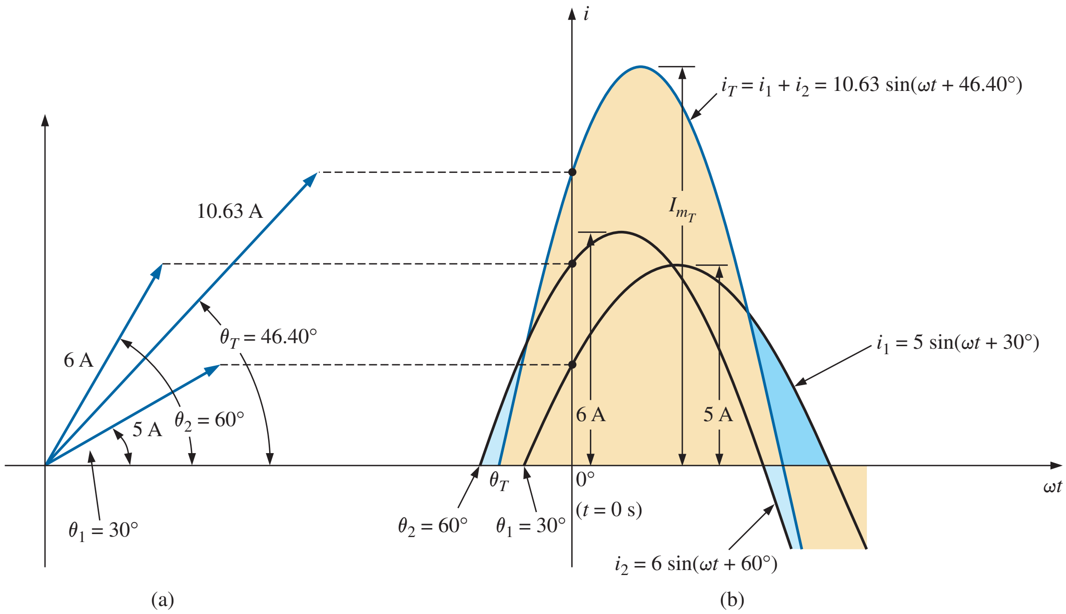

FIG. 14.73
Adding two sinusoidal currents with phase angles other than 90°.

of the functions in Fig. 14.73(b) at $t = 0$ s is determined by the rotational positions of the radius vectors in Fig. 14.73(a).

Since the rms, rather than the peak, values are used almost exclusively in the analysis of ac circuits, the phasor will now be redefined for the purposes of practicality and uniformity as having a magnitude equal to the *rms value* of the sine wave it represents. The angle associated with the phasor will remain as previously described—the phase angle.

In general, for all of the analyses to follow, the phasor form of a sinusoidal voltage or current will be

$$\mathbf{V} = V \angle\theta \quad \text{and} \quad \mathbf{I} = I \angle\theta$$

where V and I are rms values and θ is the phase angle. It should be pointed out that in phasor notation, the sine wave is always the reference, and the frequency is not represented.

Phasor algebra for sinusoidal quantities is applicable only for waveforms having the same frequency.

The use of phasor notation in the analysis of ac networks was first introduced by Professor Charles Proteus Steinmetz in 1897 (Fig. 14.74).

EXAMPLE 14.27 Convert the following from the time to the phasor domain:

Time Domain	Phasor Domain
a. $\sqrt{2}(50) \sin \omega t$	$\mathbf{50 \angle 0°}$
b. $69.6 \sin(\omega t + 72°)$	$(0.707)(69.6) \angle 72° = \mathbf{49.21 \angle 72°}$
c. $45 \cos \omega t$	$(0.707)(45) \angle 90° = \mathbf{31.82 \angle 90°}$

FIG. 14.74
Charles Proteus Steinmetz.
Courtesy of the Hall of History Foundaton, Schenectady, New York

German-American (Breslau, Germany; Yonkers and Schenectady, NY, USA)
(1865–1923)
Mathematician, Scientist, Engineer, Inventor, Professor of Electrical Engineering and Electrophysics, Union College
Department Head, General Electric Co.

Although the holder of some 200 patents and recognized worldwide for his contributions to the study of hysteresis losses and electrical transients. Charles Proteus Steinmetz is best recognized for his contribution to the study of ac networks. His "Symbolic Method of Alternating-current Calculations" provided an approach to the analysis of ac networks that removed a great deal of the confusion and frustration experienced by engineers of that day as they made the transition from dc to ac systems. His approach (from which the phasor notation of this text is premised) permitted a direct analysis of ac systems using many of the theorems and methods of analysis developed for dc systems. In 1897 he authored the epic work *Theory and Calculation of Alternating Current Phenomena,* which became the authoritative guide for practicing engineers. Dr. Steinmetz was fondly referred to as "The Doctor" at General Electric Company where he worked for some 30 years in a number of important capacities. His recognition as a "multigifted genius" is supported by the fact that he maintained active friendships with such individuals as Albert Einstein, Guglielmo Marconi (radio), and Thomas A. Edison, to name just a few. He was President of the American Institute of Electrical Engineers (AIEE) and the National Association of Corporation Schools and actively supported his local community (Schenectady) as president of the Board of Education and the Commission on Parks and City Planning.

EXAMPLE 14.28 Write the sinusoidal expression for the following phasors if the frequency is 60 Hz:

Phasor Domain	Time Domain
a. $\mathbf{I} = 10\ \angle 30°$	$i = \sqrt{2}(10)\sin(2\pi 60t + 30°)$ and $i =$ **14.14 sin (377*t* + 30°)**
b. $\mathbf{V} = 115\ \angle -70°$	$v = \sqrt{2}(115)\sin(377t - 70°)$ and $v =$ **162.6 sin (377*t* − 70°)**

EXAMPLE 14.29 Find the input voltage of the circuit in Fig. 14.75 if

$$\left.\begin{aligned} v_a &= 50\sin(377t + 30°) \\ v_b &= 30\sin(377t + 60°) \end{aligned}\right\} f = 60\text{ Hz}$$

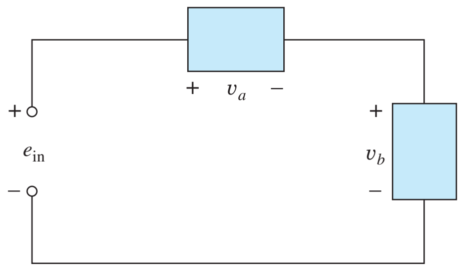

FIG. 14.75
Example 14.31.

Solution: Applying Kirchhoff's voltage law, we have

$$e_{in} = v_a + v_b$$

Converting from the time to the phasor domain yields

$$v_a = 50\sin(377t + 30°) \Rightarrow \mathbf{V}_a = 35.35\text{ V}\ \angle 30°$$
$$v_b = 30\sin(377t + 60°) \Rightarrow \mathbf{V}_b = 21.21\text{ V}\ \angle 60°$$

Converting from polar to rectangular form for addition yields

$$\mathbf{V}_a = 35.35\text{ V}\ \angle 30° = 30.61\text{ V} + j17.68\text{ V}$$
$$\mathbf{V}_b = 21.21\text{ V}\ \angle 60° = 10.61\text{ V} + j18.37\text{ V}$$

Then

$$\begin{aligned}\mathbf{E}_{in} = \mathbf{V}_a + \mathbf{V}_b &= (30.61\text{ V} + j17.68\text{ V}) + (10.61\text{ V} + j18.37\text{ V}) \\ &= 41.22\text{ V} + j36.05\text{ V}\end{aligned}$$

Converting from rectangular to polar form, we have

$$\mathbf{E}_{in} = 41.22\text{ V} + j36.05\text{ V} = 54.76\text{ V}\ \angle 41.17°$$

Converting from the phasor to the time domain, we obtain

$$\mathbf{E}_{in} = 54.76\text{ V}\ \angle 41.17° \Rightarrow e_{in} = \sqrt{2}(54.76)\sin(377t + 41.17°)$$

and $$e_{in} = \mathbf{77.43\sin(377t + 41.17°)}$$

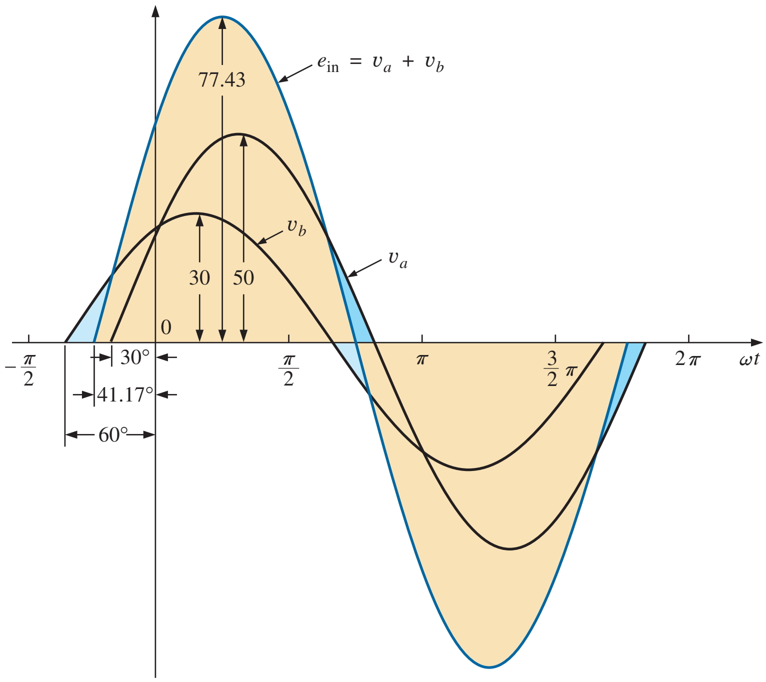

FIG. 14.76
Solution to Example 14.29.

A plot of the three waveforms is shown in Fig. 14.76. Note that at each instant of time, the sum of the two waveforms does in fact add up to e_{in}. At $t = 0$ ($\omega t = 0$), e_{in} is the sum of the two positive values, while at a value of ωt, almost midway between $\pi/2$ and π, the sum of the positive value of v_a and the negative value of v_b results in $e_{in} = 0$.

EXAMPLE 14.30 Determine the current i_2 for the network in Fig. 14.77.

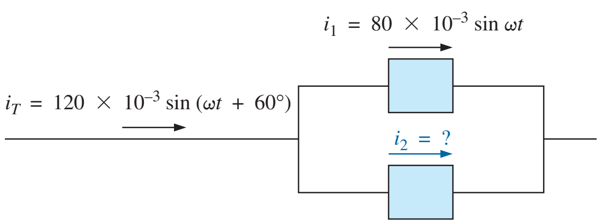

FIG. 14.77
Example 14.30.

Solution: Applying Kirchhoff's current law, we obtain

$$i_T = i_1 + i_2 \quad \text{or} \quad i_2 = i_T - i_1$$

Converting from the time to the phasor domain yields

$$i_T = 120 \times 10^{-3} \sin(\omega t + 60°) \Rightarrow 84.84 \text{ mA} \angle 60°$$
$$i_1 = 80 \times 10^{-3} \sin \omega t \Rightarrow 56.56 \text{ mA} \angle 0°$$

Converting from polar to rectangular form for subtraction yields

$$\mathbf{I}_T = 84.84 \text{ mA} \angle 60° = 42.42 \text{ mA} + j73.47 \text{ mA}$$
$$\mathbf{I}_1 = 56.56 \text{ mA} \angle 0° = 56.56 \text{ mA} + j0$$

Then

$$\mathbf{I}_2 = \mathbf{I}_T - \mathbf{I}_1$$
$$= (42.42 \text{ mA} + j73.47 \text{ mA}) - (56.56 \text{ mA} + j0)$$

and $$\mathbf{I}_2 = -14.14 \text{ mA} + j73.47 \text{ mA}$$

Converting from rectangular to polar form, we have

$$\mathbf{I}_2 = 74.82 \text{ mA} \angle 100.89°$$

Converting from the phasor to the time domain, we have

$$\mathbf{I}_2 = 74.82 \text{ mA} \angle 100.89° \Rightarrow$$
$$i_2 = \sqrt{2}(74.82 \times 10^{-3}) \sin(\omega t + 100.89°)$$

and $$\mathbf{i_2 = 105.8 \times 10^{-3} \sin(\omega t + 100.89°)}$$

A plot of the three waveforms appears in Fig. 14.78. The waveforms clearly indicate that $i_T = i_1 + i_2$.

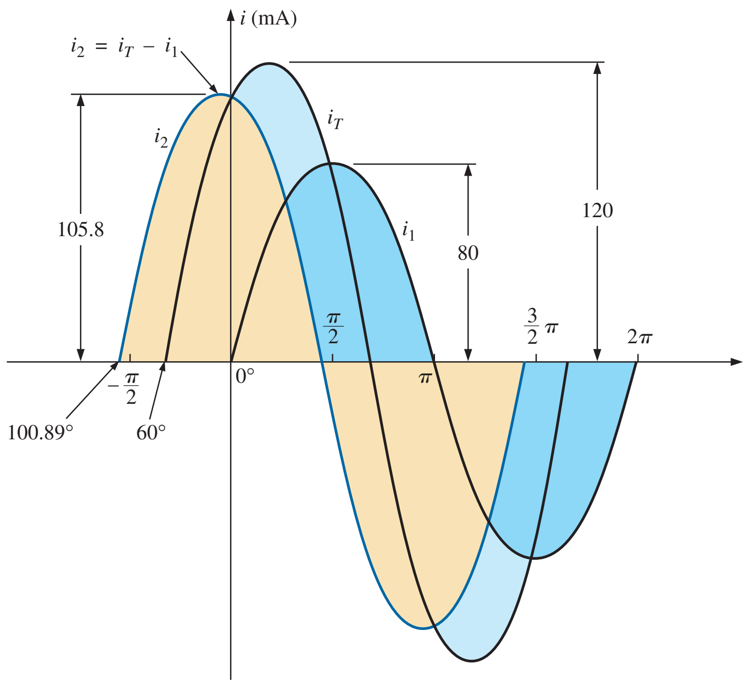

FIG. 14.78
Solution to Example 14.30.

14.13 COMPUTER ANALYSIS

PSpice

Capacitors and the ac Response The simplest of ac capacitive circuits is now analyzed to introduce the process of setting up an ac source and running an ac transient simulation. The ac source in Fig. 14.79 is obtained through **Place part** key-**SOURCE-VSIN-OK.** Change the name or value of any parameter by double-clicking on the parameter on the display or by double-clicking on the source symbol to get the **Property Editor** dialog box. Within the dialog box, set the values appearing in Fig. 14.79, and under **Display,** select **Name and Value.** After

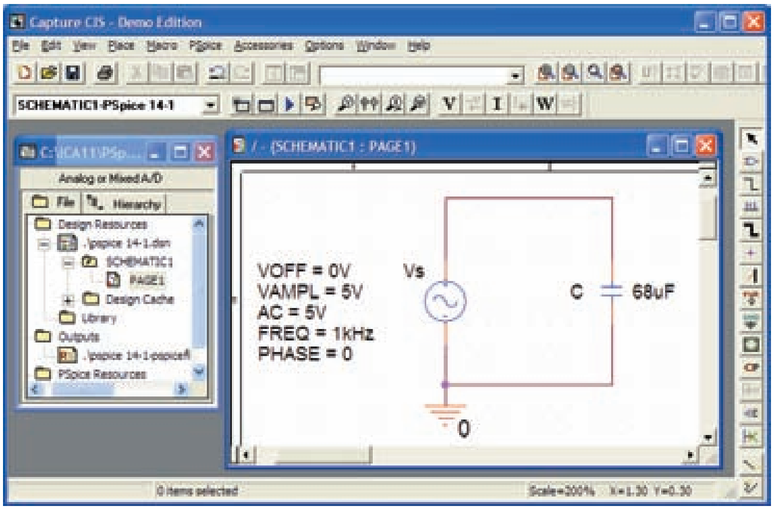

FIG. 14.79
Using PSpice to analyze the response of a capacitor to a sinusoidal ac signal.

you select **Apply** and exit the dialog box, the parameters appear as shown in Fig. 14.79.

The simulation process is initiated by selecting the **New Simulation Profile.** Under **New Simulation,** enter **PSpice 14-1** for the **Name** followed by **Create.** In the **Simulation Settings** dialog box, select **Analysis** and choose **Time Domain(Transient)** under **Analysis type.** Set the **Run to time** at 3 ms to permit a display of three cycles of the sinusoidal waveforms ($T = 1/f = 1/1000$ Hz $= 1$ ms). Leave the **Start saving data after** at 0 s, and set the **Maximum step size** at 3 ms/1000 = 3 μs. Clicking **OK** and then selecting the **Run PSpice** icon results in a plot having a horizontal axis that extends from 0 to 3 ms.

Now you must tell the computer which waveforms you are interested in. First, take a look at the applied ac source by selecting **Trace-Add Trace-V(Vs:+)** followed by **OK.** The result is the sweeping ac voltage in the bottom region of the screen in Fig. 14.80. Note that it has a peak value of 5 V, and three cycles appear in the 3 ms time frame. The current for the capacitor can be added by selecting **Trace-Add Trace** and choosing **I(C)** followed by **OK.** The resulting waveform for **I(C)** appears at a 90° phase shift from the applied voltage, with the current leading the voltage (the current has already peaked as the voltage crosses the 0 V axis). Since the peak value of each plot is in the same magnitude range, the 5 appearing on the vertical scale can be used for both. A theoretical analysis results in $X_C = 2.34\ \Omega$, and the peak value of $I_C = E/X_C = 5\text{ V}/2.34\ \Omega = 2.136$ A, as shown in Fig. 14.80.

For practice, let us obtain the curve for the power delivered to the capacitor over the same time period. First select **Plot-Add Plot to Window-Trace-Add Trace** to obtain the **Add Traces** dialog box. Then choose **V(Vs:+)**, follow it with a * for multiplication, and finish by selecting **I(C).** The result is the expression **V(Vs:+)*I(C)** of the power format: $p = vi$. Click **OK,** and the power plot at the top of Fig. 14.80 appears. Note that over the full three cycles, the area above the axis equals the area below—there is no net transfer of power over the 3 ms period. Note also that the power curve is sinusoidal (which is quite interesting) with a frequency twice that of the applied signal. Using the cursor control, we can determine that the maximum power (peak value of the sinusoidal waveform)

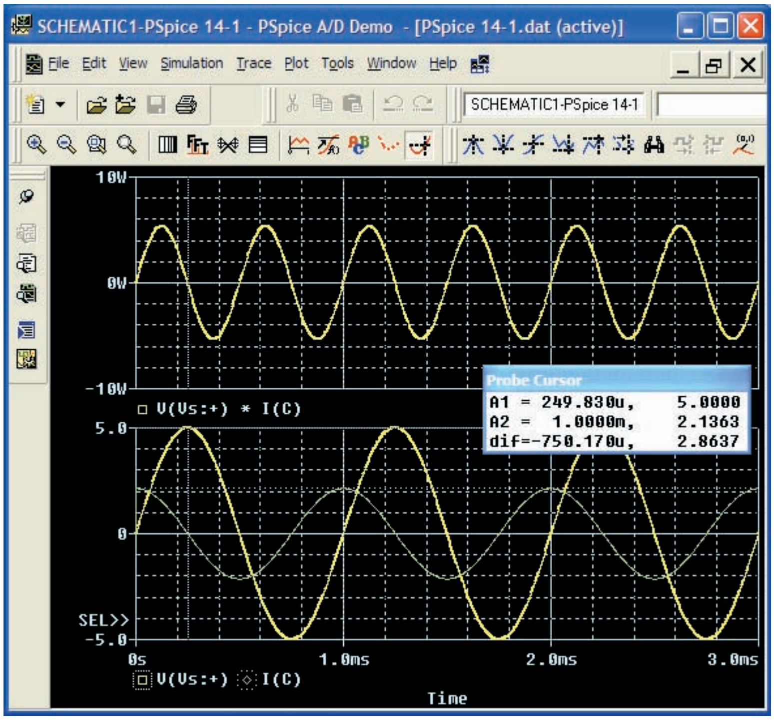

FIG. 14.80

A plot of the voltage, current, and power for the capacitor in Fig. 14.79.

is 5.34 W. The cursors, in fact, have been added to the lower curves to show the peak value of the applied sinusoid and the resulting current.

After selecting the **Toggle cursor** icon, left-click to surround the **V(Vs:+)** at the bottom of the plot with a dashed line to show that the cursor is providing the levels of that quantity. When placed at ¼ of the total period of 250 μs **(A1),** the peak value is exactly 5 V as shown in the **Probe Cursor** dialog box. Placing the cursor over the symbol next to **I(C)** at the bottom of the plot and right-clicking assigns the right cursor to the current. Placing it at exactly 1 ms (**A2**) results in a peak value of 2.136 A to match the solution above. To further distinguish between the voltage and current waveforms, the color and the width of the lines of the traces were changed. Place the cursor right on the plot line and right-click. The **Properties** option appears. When **Properties** is selected, a **Trace Properties** dialog box appears in which the yellow color can be selected and the width widened to improve the visibility on the black background. Note that yellow was chosen for **Vs** and green for **I(C).** Note also that the axis and the grid have been changed to a more visible color using the same procedure.

Multisim

Since PSpice reviewed the response of a capacitive element to an ac voltage, Multisim repeats the analysis for an inductive element. The ac voltage source was derived from the **Sources** parts bin as described in Chapter 13 with the values appearing in Fig. 14.81 set in the **AC Voltage** dialog box. Since the transient response of Multisim is limited to a plot of voltage versus time, a plot of the current of the circuit requires the addition of a resistor of 1 Ω in series with the inductive element. The magnitude of the current through the resistor and, of course, the series inductor is then determined by

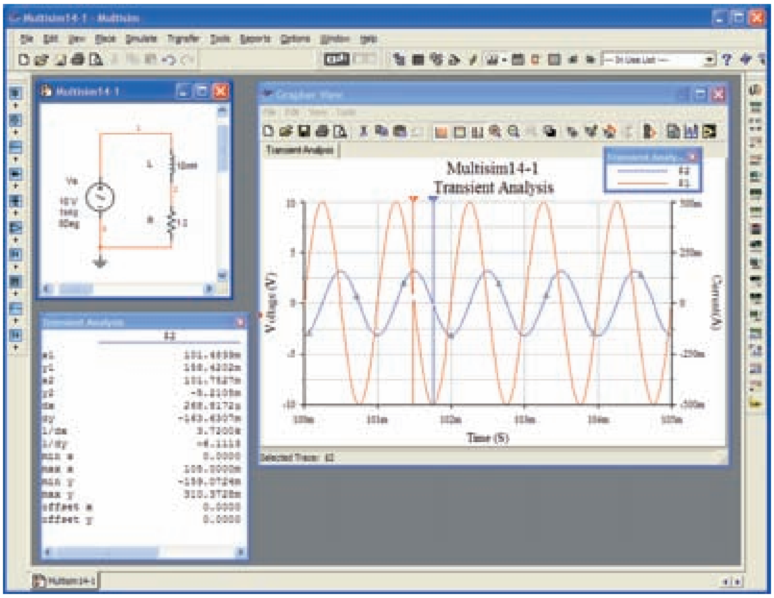

FIG. 14.81
Using Multisim to review the response of an inductive element to a sinusoidal ac signal.

$$|i_R| = \left|\frac{\upsilon_R}{R}\right| = \left|\frac{\upsilon_R}{1\ \Omega}\right| = |\upsilon_R| = |i_L|$$

revealing that the current has the same peak value as the voltage across the resistor due to the division by 1. When viewed on the graph, it can simply be considered a plot of the current. In actuality, all inductors require a series resistance, so the 1 Ω resistor serves an important dual purpose. The 1 Ω resistance is also so small compared to the reactance of the coil at the 1 kHz frequency that its effect on the total impedance or voltage across the coil can be ignored.

Once the circuit has been constructed, the sequence **Simulate-Analyses-Transient Analysis** results in a **Transient Analysis** dialog box in which the **Start time** is set at 0 s and the **End time** at 105 ms. The 105 ms was set as the **End time** to give the network 100 ms to settle down in its steady-state mode and 5 ms for five cycles in the output display. The **Minimum number of time points** was set at 10,000 to ensure a good display for the rapidly changing waveforms.

Next the **Output variables** heading was chosen within the dialog box, and nodes **1** and **2** were moved from the **Variables in Circuit** to **Selected variables for analysis** using the **Add** option. Choosing **Simulate** results in a waveform that extends from 0 s to 105 ms. Even though we plan to save only the response that occurs after 100 ms, the computer is unaware of our interest, and it plots the response for the entire period. This is corrected by selecting the **Properties** keypad in the toolbar at the top of the graph (it looks like a tag and pencil) to obtain the **Graph Properties** dialog box. Selecting **Bottom Axis** permits setting the **Range** from a **Minimum of 0.100s=100ms** to a **Maximum of 0.105s=105ms.** Click **OK,** and the time period of Fig. 14.81 is displayed. The grid structure is added by selecting the **Show/Hide Grid** keypad, and the color associated with each nodal voltage is displayed if we choose the **Show/Hide Legend** key next to it.

The scale for the plot of i_L can be improved by first going to **Traces** and setting the **Trace** to the number **2** representing the voltage across the 1 Ω resistor. When **2** is selected, the **Color** displayed automatically changes to blue. In the **Y Range,** select **Right Axis** followed by **OK.** Then select the **Right Axis** heading, and enter **Current(A)** for the **Label,** enable **Axis,** change the **Pen Size** to 1, and change the **Range** from −500 mA to +500 mA. Finally, set the **Total Ticks** at 8 with **Minor Ticks** at 2 to match the **Left Axis,** and leave the box with an **OK.** The plot in Fig. 14.81 results. Take immediate note of the new axis on the right and the **Current(A)** label. We can now see that the current has a peak of about 160 mA. For more detail on the peak values, click on the **Show/Hide Cursors** keypad on the top toolbar. A **Transient Analysis** dialog box appears with a **1** and a red line to indicate that it is working on the full source voltage at node **1.** To switch to the current curve (the blue curve), bring the cursor to any point on the blue curve and left-click. A blue line and the number **2** appear at the heading of the **Transient Analysis** dialog box. Clicking on the **1** in the small inverted arrow at the top allows you to drag the vertical red line to any horizontal point on the graph. As shown in Fig. 14.81, when the cursor is set on 101.5 ms (**x1**), the peak value of the current curve is 159.05 mA **(y1)**. A second cursor appears in blue with a number **2** in the inverted arrowhead that can also be moved with a left click on the number **2** at the top of the line. If set at 101.75 ms **(x2)**, it has a minimum value of −5.18 mA **(y2)**, the smallest value available for the calculated data points. Note that the difference between horizontal time values **dx** = 252 μs = 0.25 ms which is ¼ of the period of the wave (at 1 ms).

PROBLEMS

SECTION 14.2 Derivative

1. Plot the following waveform versus time showing one clear, complete cycle. Then determine the derivative of the waveform using Eq. (14.1), and sketch one complete cycle of the derivative directly under the original waveform. Compare the magnitude of the derivative at various points versus the slope of the original sinusoidal function.

$$v = 1 \sin 3.14t$$

2. Repeat Problem 1 for the following sinusoidal function, and compare results. In particular, determine the frequency of the waveforms of Problems 1 and 2, and compare the magnitude of the derivative.

$$v = 1 \sin 15.71t$$

3. What is the derivative of each of the following sinusoidal expressions?
   a. $10 \sin 377t$ **b.** $0.6 \sin(754t + 20°)$
   c. $\sqrt{2}\, 20 \sin(157t - 20°)$ **d.** $-200 \sin(t + 180°)$

SECTION 14.3 Response of Basic *R, L,* and *C* Elements to a Sinusoidal Voltage or Current

4. The voltage across a 5 Ω resistor is as indicated. Find the sinusoidal expression for the current. In addition, sketch the v and i sinusoidal waveforms on the same axis.
   a. $150 \sin 200t$ **b.** $30 \sin(377t + 20°)$
   c. $40 \cos(\omega t + 10°)$ **d.** $-80 \sin(\omega t + 40°)$

5. The current through a 7 kΩ resistor is as indicated. Find the sinusoidal expression for the voltage. In addition, sketch the v and i sinusoidal waveforms on the same axis.
   a. $0.1 \sin 1000t$
   b. $2 \times 10^{-3} \sin(400t - 120°)$
   c. $6 \times 10^{-6} \cos(\omega t - 2°)$
   d. $-0.004 \cos(\omega t + 90°)$

6. Determine the inductive reactance (in ohms) of a 2 H coil for
   a. dc
   and for the following frequencies:
   b. 10 Hz **c.** 60 Hz
   d. 2000 Hz **e.** 100,000 Hz

7. Determine the inductance of a coil that has a reactance of
   a. 20 Ω at $f = 2$ Hz.
   b. 1000 Ω at $f = 60$ Hz.
   c. 5280 Ω at $f = 500$ Hz.

8. Determine the frequency at which a 10 H inductance has the following inductive reactances:
   a. 100 Ω **b.** 3770 Ω
   c. 15.7 kΩ **d.** 243 Ω

9. The current through a 20 Ω inductive reactance is given. What is the sinusoidal expression for the voltage? Sketch the v and i sinusoidal waveforms on the same axis.
   a. $i = 5 \sin \omega t$
   b. $i = 40 \times 10^{-3} \sin(\omega t + 60°)$
   c. $i = -6 \sin(\omega t - 30°)$
   d. $i = 3 \cos(\omega t + 10°)$

10. The current through a. 0.1 H coil is given. What is the sinusoidal expression for the voltage?
a. $10 \sin 100t$
b. $6 \times 10^{-3} \sin 377t$
c. $5 \times 10^{-6} \sin(400t + 20°)$
d. $-4 \cos(20t - 70°)$

11. The voltage across a 50 Ω inductive reactance is given. What is the sinusoidal expression for the current? Sketch the υ and i sinusoidal waveforms on the same set of axes.
a. $120 \sin \omega t$ **b.** $30 \sin(\omega t + 20°)$
c. $40 \cos(\omega t + 10°)$ **d.** $-80 \sin(377t + 40°)$

12. The voltage across a 0.2 H coil is given. What is the sinusoidal expression for the current?
a. $1.5 \sin 60t$
b. $16 \times 10^{-3} \sin(10t + 2°)$
c. $-4.8 \sin(0.05t + 50°)$
d. $9 \times 10^{-3} \cos(377t + 360°)$

13. Determine the capacitive reactance (in ohms) of a 5 μF capacitor for
a. dc
and for the following frequencies:
b. 60 Hz **c.** 120 Hz
d. 2 kHz **e.** 2 MHz

14. Determine the capacitance in microfarads if a capacitor has a reactance of
a. 250 Ω at $f = 60$ Hz.
b. 55 Ω at $f = 312$ Hz.
c. 10 Ω at $f = 25$ Hz.

15. Determine the frequency at which a 50 μF capacitor has the following capacitive reactances:
a. 100 Ω **b.** 684 Ω
c. 342 Ω **d.** 2000 Ω

16. The voltage across a 2.5 Ω capacitive reactance is given. What is the sinusoidal expression for the current? Sketch the υ and i sinusoidal waveforms on the same set of axes.
a. $120 \sin \omega t$ **b.** $0.4 \sin(\omega t + 20°)$
c. $8 \cos(\omega t + 10°)$ **d.** $-70 \sin(\omega t + 40°)$

17. The voltage across a 1 μF capacitor is given. What is the sinusoidal expression for the current?
a. $30 \sin 200t$ **b.** $60 \times 10^{-3} \sin 377t$
c. $-120 \sin(374t + 30°)$ **d.** $70 \cos(800t - 20°)$

18. The current through a 10 Ω capacitive reactance is given. Write the sinusoidal expression for the voltage. Sketch the υ and i sinusoidal waveforms on the same set of axes.
a. $i = 50 \times 10^{-3} \sin \omega t$
b. $i = 2 \times 10^{-6} \sin(\omega t + 60°)$
c. $i = -6 \sin(\omega t - 30°)$
d. $i = 3 \cos(\omega t + 10°)$

19. The current through a 0.5 μF capacitor is given. What is the sinusoidal expression for the voltage?
a. $0.20 \sin 300t$ **b.** $8 \times 10^{-3} \sin 377t$
c. $60 \times 10^{-3} \cos 754t$ **d.** $0.08 \sin(1600t - 80°)$

***20.** For the following pairs of voltages and currents, indicate whether the element involved is a capacitor, an inductor, or a resistor, and the value of C, L or R if sufficient data are given:
a. $\upsilon = 550 \sin(377t + 50°)$
$i = 11 \sin(377t - 40°)$
b. $\upsilon = 36 \sin(754t - 80°)$
$i = 4 \sin(754t - 170°)$
c. $\upsilon = 10.5 \sin(\omega t - 13°)$
$i = 1.5 \sin(\omega t - 13°)$

***21.** Repeat Problem 20 for the following pairs of voltages and currents:
a. $\upsilon = 2000 \sin \omega t$
$i = 5 \cos \omega t$
b. $\upsilon = 80 \sin(157t + 150°)$
$i = 2 \sin(157t + 60°)$
c. $\upsilon = 35 \sin(\omega t - 20°)$
$i = 7 \cos(\omega t - 110°)$

SECTION 14.4 Frequency Response of the Basic Elements

22. Plot X_L versus frequency for a 5 mH coil using a frequency range of zero to 100 kHz on a linear scale.

23. Plot X_C versus frequency for a 1 μF capacitor using a frequency range of zero to 10 kHz on a linear scale.

24. At what frequency will the reactance of a 1 μF capacitor equal the resistance of a 2 kΩ resistor?

25. The reactance of a coil equals the resistance of a 10 kΩ resistor at a frequency of 5 kHz. Determine the inductance of the coil.

26. Determine the frequency at which a 1 μF capacitor and a 10 mH inductor will have the same reactance.

27. Determine the capacitance required to establish a capacitive reactance that will match that of a 2 mH coil at a frequency of 50 kHz.

SECTION 14.5 Average Power and Power Factor

28. Find the average power loss in watts for each set in Problem 20.

29. Find the average power loss in watts for each set in Problem 21.

***30.** Find the average power loss and power factor for each of the circuits whose input current and voltage are as follows:
a. $\upsilon = 60 \sin(\omega t + 30°)$
$i = 15 \sin(\omega t + 60°)$
b. $\upsilon = -50 \sin(\omega t - 20°)$
$i = -2 \sin(\omega t - 20°)$
c. $\upsilon = 50 \sin(\omega t + 80°)$
$i = 3 \cos(\omega t - 20°)$
d. $\upsilon = 75 \sin(\omega t - 5°)$
$i = 0.08 \sin(\omega t + 35°)$

31. If the current through and voltage across an element are $i = 8 \sin(\omega t + 40°)$ and $\upsilon = 48 \sin(\omega t + 40°)$, respectively, compute the power by I^2R, $(V_mI_m/2) \cos \theta$, and $VI \cos \theta$, and compare answers.

32. A circuit dissipates 100 W (average power) at 150 V (effective input voltage) and 2 A (effective input current). What is the power factor? Repeat if the power is 0 W; 300 W.

***33.** The power factor of a circuit is 0.5 lagging. The power delivered in watts is 500. If the input voltage is $50 \sin(\omega t + 10°)$, find the sinusoidal expression for the input current.

34. In Fig. 14.82, $e = 30 \sin(377t + 20°)$.
  a. What is the sinusoidal expression for the current?
  b. Find the power loss in the circuit.
  c. How long (in seconds) does it take the current to complete six cycles?

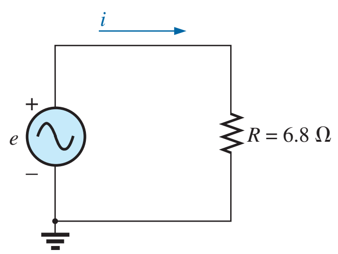

FIG. 14.82
Problem 34.

35. In Fig. 14.83, $e = 100 \sin(314t + 60°)$.
  a. Find the sinusoidal expression for i.
  b. Find the value of the inductance L.
  c. Find the average power loss by the inductor.

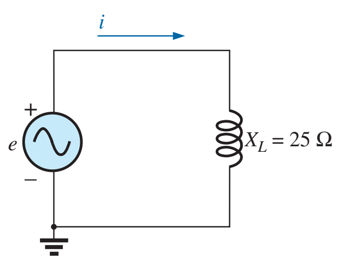

FIG. 14.83
Problem 35.

36. In Fig. 14.84, $i = 30 \times 10^{-3} \sin(377t - 20°)$.
  a. Find the sinusoidal expression for e.
  b. Find the value of the capacitance C in microfarads.
  c. Find the average power loss in the capacitor.

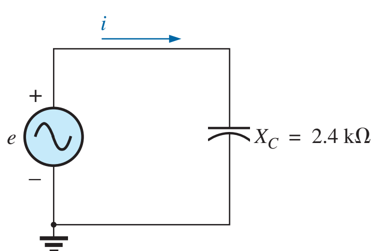

FIG. 14.84
Problem 36.

***37.** For the network in Fig. 14.85 and the applied signal:
  a. Determine i_1 and i_2.
  b. Find i_s.

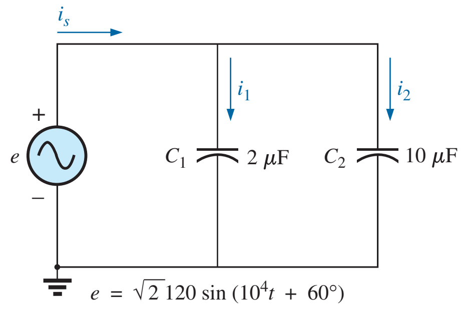

FIG. 14.85
Problem 37.

***38.** For the network in Fig. 14.86 and the applied source:
  a. Determine the source voltage v_s.
  b. Find the currents i_1 and i_2.

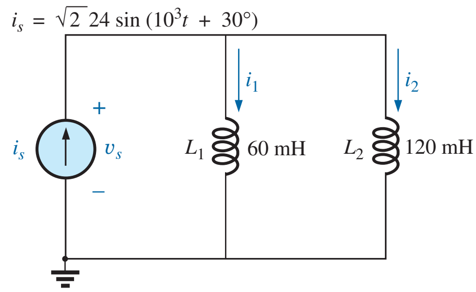

FIG. 14.86
Problem 38.

SECTION 14.9 Conversion between Forms

39. Convert the following from rectangular to polar form:
  a. $4 + j3$ **b.** $2 + j2$
  c. $6 + j16$ **d.** $100 + j1000$
  e. $1000 + j400$ **f.** $0.001 + j0.0065$
  g. $7.6 - j9$ **h.** $-8 - j4$
  i. $-15 - j60$ **j.** $+78 - j65.3$
  k. $-2400 + j3600$ **l.** $5 \times 10^{-3} - j25 \times 10^{-3}$

40. Convert the following from polar to rectangular form:
  a. $6 \angle 30°$ **b.** $40 \angle 80°$
  c. $7400 \angle 70°$ **d.** $4 \times 10^{-4} \angle 8°$
  e. $0.04 \angle 90°$ **f.** $0.0093 \angle 42°$
  g. $65 \angle 150°$ **h.** $1.2 \angle 135°$
  i. $500 \angle 200°$ **j.** $6320 \angle -35°$
  k. $7.52 \angle -125°$ **l.** $8 \times 10^{-3} \angle 210°$

41. Convert the following from rectangular to polar form:
  a. $1 + j15$ **b.** $60 + j5$
  c. $0.01 + j0.3$ **d.** $100 - j200$
  e. $-5.6 + j86$ **f.** $-2.7 - j38.6$

42. Convert the following from polar to rectangular form:
  a. $13 \angle 5°$ **b.** $160 \angle 87°$
  c. $7 \times 10^{-6} \angle 2°$ **d.** $8.7 \angle 177°$
  e. $76 \angle -4°$ **f.** $396 \angle +265°$

SECTION 14.10 Mathematical Operations with Complex Numbers

Perform the following operations.

43. Addition and subtraction (express your answers in rectangular form):
- **a.** $(4.2 + j6.8) + (7.6 + j0.2)$
- **b.** $(142 + j7) + (9.8 + j42) + (0.1 + j0.9)$
- **c.** $(4 \times 10^{-6} + j76) + (7.2 \times 10^{-7} - j5)$
- **d.** $(9.8 + j6.2) - (4.6 + j4.6)$
- **e.** $(167 + j243) - (-42.3 - j68)$
- **f.** $(-36.0 + j78) - (-4 - j6) + (10.8 - j72)$
- **g.** $6\angle 20° + 8\angle 80°$
- **h.** $42\angle 45° + 62\angle 60° - 70\angle 120°$

44. Multiplication [express your answers in rectangular form for parts (a) through (d), and in polar form for parts (e) through (h)]:
- **a.** $(2 + j3)(6 + j8)$
- **b.** $(7.8 + j1)(4 + j2)(7 + j6)$
- **c.** $(0.002 + j0.006)(-4 + j8)$
- **d.** $(400 - j200)(-0.01 - j0.5)(-1 + j3)$
- **e.** $(2\angle 60°)(4\angle -40°)$
- **f.** $(6.9\angle 8°)(7.2\angle -72°)$
- **g.** $(0.002\angle 120°)(0.5\angle 200°)(40\angle +80°)$
- **h.** $(540\angle -20°)(-5\angle 180°)(6.2\angle 0°)$

45. Division (express your answer in polar form):
- **a.** $(42\angle 10°)/(7\angle 60°)$
- **b.** $(0.006\angle 120°)/(30\angle +60°)$
- **c.** $(4360\angle -20°)/(40\angle -210°)$
- **d.** $(650\angle -80°)/(8.5\angle 360°)$
- **e.** $(8 + j8)/(2 + j2)$
- **f.** $(8 + j42)/(-6 - j4)$
- **g.** $(0.05 + j0.25)/(8 - j60)$
- **h.** $(-4.5 - j6)/(0.1 - j0.8)$

***46.** Perform the following operations (express your answers in rectangular form):
- **a.** $\dfrac{(4 + j3) + (6 - j8)}{(3 + j3) - (2 + j3)}$
- **b.** $\dfrac{8\angle 60°}{(2\angle 0°) + (100 + j400)}$
- **c.** $\dfrac{(6\angle 20°)(120\angle -40°)(3 + j8)}{2\angle -30°}$
- **d.** $\dfrac{(0.4\angle 60°)^2(300\angle 40°)}{3 + j9}$
- **e.** $\left(\dfrac{1}{(0.02\angle 10°)^2}\right)\left(\dfrac{2}{j}\right)^3\left(\dfrac{1}{6^2 - j\sqrt{900}}\right)$

***47. a.** Determine a solution for x and y if

$$(x + j4) + (3x + jy) - j7 = 16\angle 0°$$

b. Determine x if

$$(10\angle 20°)(x\angle -60°) = 30.64 - j25.72$$

c. Determine a solution for x and y if

$$(5x + j10)(2 - jy) = 90 - j70$$

d. Determine θ if

$$\frac{80\angle 0°}{20\angle \theta} = 3.464 - j2$$

SECTION 14.12 Phasors

48. Express the following in phasor form:
- **a.** $\sqrt{2}(160)\sin(\omega t + 30°)$
- **b.** $\sqrt{2}(25 \times 10^{-3})\sin(157t - 40°)$
- **c.** $100\sin(\omega t - 90°)$
- **d.** $20\sin(377t + 0°)$
- **e.** $6 \times 10^{-6}\cos \omega t$
- **f.** $3.6 \times 10^{-6}\cos(754t - 20°)$

49. Express the following phasor currents and voltages as sine waves if the frequency is 60 Hz:
- **a.** $\mathbf{I} = 40\text{ A}\angle 20°$
- **b.** $\mathbf{V} = 120\text{ V}\angle 10°$
- **c.** $\mathbf{I} = 8 \times 10^{-3}\text{ A}\angle 120°$
- **d.** $\mathbf{V} = 5\text{ V}\angle 90°$
- **e.** $\mathbf{I} = 1200\text{ A}\angle -50°$
- **f.** $\mathbf{V} = \dfrac{6000}{\sqrt{2}}\text{ V}\angle -180°$

50. For the system in Fig. 14.87, find the sinusoidal expression for the unknown voltage v_a if

$$e_{in} = 60\sin(377t + 20°)$$
$$v_b = 20\sin(377t - 20°)$$

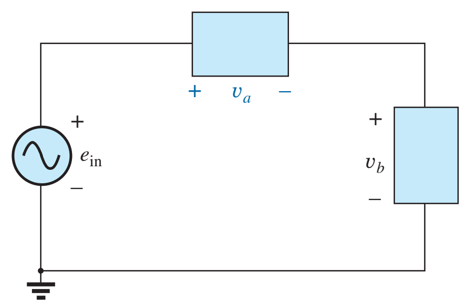

FIG. 14.87
Problem 50.

51. For the system in Fig. 14.88, find the sinusoidal expression for the unknown current i_1 if

$$i_s = 20 \times 10^{-6}\sin(\omega t + 60°)$$
$$i_2 = 6 \times 10^{-6}\sin(\omega t - 30°)$$

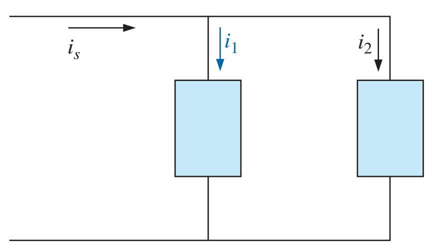

FIG. 14.88
Problem 51.

52. Find the sinusoidal expression for the applied voltage e for the system in Fig. 14.89 if

$$\upsilon_a = 60 \sin(\omega t + 30°)$$
$$\upsilon_b = 30 \sin(\omega t + 60°)$$
$$\upsilon_c = 40 \sin(\omega t + 120°)$$

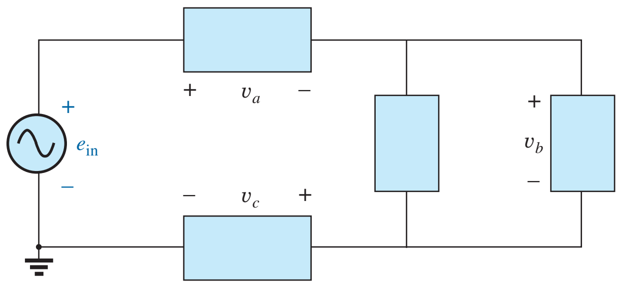

FIG. 14.89
Problem 52.

53. Find the sinusoidal expression for the current i_s for the system in Fig. 14.90 if

$$i_1 = 6 \times 10^{-3} \sin(377t + 180°)$$
$$i_2 = 8 \times 10^{-3} \sin(377t - 180°)$$
$$i_3 = 2i_2$$

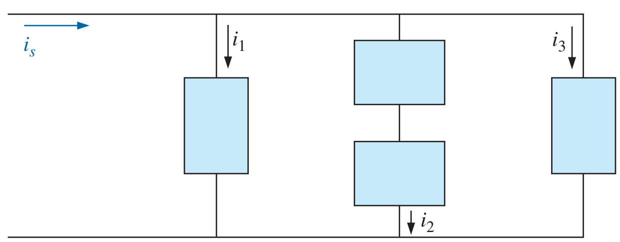

FIG. 14.90
Problem 53.

SECTION 14.13 Computer Analysis

PSpice or Multisim

54. Plot i_c and υ_c versus time for the network in Fig. 14.79 for two cycles if the frequency is 0.2 kHz.

55. Plot the magnitude and phase angle of the current i_C versus frequency (100 Hz to 100 kHz) for the network in Fig. 14.79.

56. Plot the total impedance of the configuration in Fig. 14.27 versus frequency (100 kHz to 100 MHz) for the following parameter values: $C = 0.1\ \mu F$, $L_s = 0.2\ \mu H$, $R_s = 2\ M\Omega$, and $R_p = 100\ M\Omega$. For what frequency range is the capacitor "capacitive"?

GLOSSARY

Average or **real power** The power delivered to and dissipated by the load over a full cycle.

Complex conjugate A complex number defined by simply changing the sign of an imaginary component of a complex number in the rectangular form.

Complex number A number that represents a point in a two-dimensional plane located with reference to two distinct axes. It defines a vector drawn from the origin to that point.

Derivative The instantaneous rate of change of a function with respect to time or another variable.

Leading and **lagging power factors** An indication of whether a network is primarily capacitive or inductive in nature. Leading power factors are associated with capacitive networks, and lagging power factors with inductive networks.

Phasor A radius vector that has a constant magnitude at a fixed angle from the positive real axis and that represents a sinusoidal voltage or current in the vector domain.

Phasor diagram A "snapshot" of the phasors that represent a number of sinusoidal waveforms at $t = 0$.

Polar form A method of defining a point in a complex plane that includes a single magnitude to represent the distance from the origin, and an angle to reflect the counterclockwise distance from the positive real axis.

Power factor (F_p) An indication of how reactive or resistive an electrical system is. The higher the power factor, the greater the resistive component.

Reactance The opposition of an inductor or a capacitor to the flow of charge that results in the continual exchange of energy between the circuit and magnetic field of an inductor or the electric field of a capacitor.

Reciprocal A format defined by 1 divided by the complex number.

Rectangular form A method of defining a point in a complex plane that includes the magnitude of the real component and the magnitude of the imaginary component, the latter component being defined by an associated letter *j*.